A New Method for Estimating Maximum Power Transfer and Voltage Stability Margins to Mitigate the Risk of Voltage Collapse

Bernie Lesieutre
Dan Molzahn
University of Wisconsin-Madison

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Voltage Stability

Static:
- Loading, Power Flows
- Local Reactive Power Support

Dynamic:
- Component Controls
- Network Controls
A common voltage stability margin measures the distance from a post-contingency operating point to the “nose point” on a power-voltage curve.
OVERVIEW: Static Voltage Stability Margin

Voltage, $V$

Margin (negative)

It’s even possible that a normal power flow solution won’t exist, post contingency.
1. We don’t know the values on the curve.
2. We don’t know whether a post-contingency operating point exists!
Typical Approach

• Run post-contingency power flow. This may or may not converge.

• If successful, determine nose point: use a sequence of power flows with increasing load, or a continuation power flow.
Our Approach

Cast as an optimization problem:

• Minimize the controlled voltages while a solution exists. (Claim: a solution exists.)

• Exploit the quadratic nature of the power flow equations to directly obtain Traditional Voltage Stability Margin even when the margin is negative.
Advantages

• Eliminates need for repeated solutions (multiple power flows, continuation power flows)
• Often offers provably *globally* optimal results
• Works when the margin is negative, i.e. when there isn’t a solution.
OVERVIEW: Static Voltage Stability Margin

Voltage, $V$

$V_0$

$V_{opt}$

Margin

$P_{max}$

$P_0$

Optimal Solution

$$\frac{P_{max}}{P_0} = \left( \frac{V_0}{V_{opt}} \right)^2$$
Power Flow Equations: Review

Electrical Network
\[ I = YV \]
\[ Y = G + jB \]

Current “Injection” equations in Rectangular Coordinates:

\[
\begin{bmatrix}
(I_{D1} + jI_{Q1}) \\
\vdots \\
(I_{DN} + jI_{QN})
\end{bmatrix}
\begin{bmatrix}
G + jB
\end{bmatrix}
\begin{bmatrix}
(V_{D1} + jV_{Q1}) \\
\vdots \\
(V_{DN} + jV_{QN})
\end{bmatrix}
\]
Power Flow Equations: Review

Electrical Network
\[ I = YV \]
\[ Y = G + jB \]

Power "Injection" equations in Rectangular Coordinates:

\[
\begin{bmatrix}
P_1 + jQ_1 \\
\vdots \\
P_N + jQ_N
\end{bmatrix}
= 
\begin{bmatrix}
V_{D1} + jV_{Q1} \\
\vdots \\
V_{DN} + jV_{QN}
\end{bmatrix}
\begin{bmatrix}
G - jB \\
\vdots \\
(V_{DN} - jV_{QN})
\end{bmatrix}
\begin{bmatrix}
(V_{D1} - jV_{Q1}) \\
\vdots \\
(V_{DN} - jV_{QN})
\end{bmatrix}
\]
Power Flow Equations

The power flow equations are **quadratic** in voltage variables:

\[
P_i = V_{Di} \sum_{k=1}^{N} (G_{ik} V_{Dk} - B_{ik} V_{Qk}) + V_{Qi} \sum_{k=1}^{N} (B_{ik} V_{Dk} + G_{ik} V_{Qk})
\]

\[
Q_i = -V_{Di} \sum_{k=1}^{N} (B_{ik} V_{Dk} + G_{ik} V_{Qk}) + V_{Qi} \sum_{k=1}^{N} (G_{ik} V_{Dk} - B_{ik} V_{Qk})
\]

For reference, power engineers almost always express these equations in voltage polar coordinates:

\[
P_k = V_k \sum_{i=1}^{n} V_i (G_{ik} \cos (\delta_k - \delta_i) + B_{ik} \sin (\delta_k - \delta_i))
\]

\[
Q_k = V_k \sum_{i=1}^{n} V_i (G_{ik} \sin (\delta_k - \delta_i) - B_{ik} \cos (\delta_k - \delta_i))
\]
The “controlled voltages” are the problem-specified slack and generator voltage magnitudes.

Locking the controlled voltages in constant proportion, and allowing them to scale by \( \alpha \), we claim a power solution exists for any power flow injection profile. (subject to the unimportant small print.)

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<tr>
<th>Bus Type</th>
<th>Specified</th>
<th>Calculated</th>
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<tbody>
<tr>
<td>PQ (load)</td>
<td>( P, Q )</td>
<td>( V, \delta )</td>
</tr>
<tr>
<td>PV (generator)</td>
<td>( P, V )</td>
<td>( Q, \delta )</td>
</tr>
<tr>
<td>Slack</td>
<td>( V, \delta = 0^\circ )</td>
<td>( P, Q )</td>
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</table>
Optimization Problem

- Modify the power flow formulation to \((\dagger)\)
- Slack bus voltage magnitude unconstrained
- PV bus voltage magnitudes scale with slack bus voltage
- Minimize slack bus voltage

\[
V_{\text{slack}}^{\text{opt}} = \min \quad V_{\text{slack}}
\]

subject to

\[
P_k = V_k \sum_{i=1}^{n} V_i \left( G_{ik} \cos (\delta_k - \delta_i) + B_{ik} \sin (\delta_k - \delta_i) \right) \quad \forall k \in \{\mathcal{PQ}, \mathcal{PV}\}
\]

\[
Q_k = V_k \sum_{i=1}^{n} V_i \left( G_{ik} \sin (\delta_k - \delta_i) - B_{ik} \cos (\delta_k - \delta_i) \right) \quad \forall k \in \mathcal{PQ}
\]

\[
V_k = \alpha_k V_{\text{slack}} \quad \forall k \in \mathcal{PV}
\]
This optimization problem can be solved many different ways…

We’ve been using the convex relaxation formulation for the power flow equations (Lavaei, Low) because we really want (provably) the minimum solution.

- The problem has a feasible solution
- The optimization using the convex relaxation can be solved for a global minimum (and hopefully a feasible power flow solution).
Relaxed Problem Formulation

In Rectangular coordinates, define

\[ x = [V_{d1} \ldots V_{dN} V_{q1} \ldots V_{qN}]^T \]

Then, power flow equations can be written in the form

\[ P_k = c_k^T (x \; x^T) c_k = tr(C_k W) \]
\[ Q_k = \bar{c}_k^T (x \; x^T) \bar{c}_k = tr(\bar{C}_k W) \]
\[ V_k^2 = tr(M_k W) \]

where

\[ W = x \; x^T \]

Which is a rank one matrix by construction.
Relaxed Problem Formulation

The convex relaxation is introduced by relaxing the rank of $W$. With that, we pose the following convex optimization problem:

$$V_{slack}^{opt} = \min V_{slack}$$

subject to

$$P_k = tr(Y_k W)$$

$$Q_k = tr(\bar{Y}_k W)$$

$$\alpha_k^2 V_{slack}^2 = tr(M_k W)$$

$$W \succeq 0 \quad W \text{ is positive semi-definite}$$
Semidefinite Relaxation Example

\[ x = \begin{bmatrix} V_{d1} & V_{d2} & \ldots & V_{dn} & V_{q1} & V_{q2} & \ldots & V_{qn} \end{bmatrix}^T \]

\[ M_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 \end{bmatrix} \]

\[ W = xx^T = \begin{bmatrix} V_{d1}^2 & V_{d1}V_{d2} & \ldots & V_{d1}V_{q1} & \ldots & V_{d1}V_{qn} \\ V_{d1}V_{d2} & V_{d2}^2 & \ldots & V_{d2}V_{q1} & \ldots & V_{d2}V_{qn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ V_{d1}V_{q1} & V_{d2}V_{q1} & \ldots & V_{q1}^2 & \ldots & V_{q1}V_{qn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ V_{d1}V_{qn} & V_{d2}V_{qn} & \ldots & V_{q1}V_{qn} & \ldots & V_{qn}^2 \end{bmatrix} \]

\[ = V_{d1}^2 + V_{q1}^2 = V_1^2 \]
Semidefinite Relaxation Example

\[ \begin{align*}
\text{trace} & \left[ \begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix} \right] \\
& = W_{11} + W_{n+1, n+1}
\end{align*} \]
Bonus result concerning solution to the power flow equations

- The existence of a power flow solution requires

\[ V_{\text{slack}}^{\text{opt}} \leq V_0 \] (Specified slack bus voltage)

- Necessary, but not sufficient, condition for existence

- Conversely, no solution exists if \( V_{\text{slack}}^{\text{opt}} > V_0 \)

- Sufficient, but not necessary, condition for non-existence
Controlled Voltage Margin

- A controlled voltage margin to the solvability boundary

\[ \sigma = \frac{V_0}{V_{\text{min}}^{\text{slack}}} \]

- Upper bound (non-conservative)

- No power flow solution exists for \( \sigma < 1 \)

- Increasing the slack bus voltage (with proportional increases in PV bus voltages) by at least \( \frac{1}{\sigma} \) is required for solution.
Power Injection Margin

- Uniformly scaling all power injections scales

\[ (V_{\text{slack}})^2 = f(P_{\text{inj}} + jQ_{\text{inj}}) \]

- Uniformly scale power injections until

\[ \eta (V_{\text{slack}})^2 = f(\eta (P_{\text{inj}} + jQ_{\text{inj}})) \]

- Corresponding \( \eta \) gives a power flow voltage stability margin in the direction of uniformly increasing power injections at constant power factor.

- \( \eta < 1 \) indicates that no solution exists for the original power flow problem
Power Injection Margin

The power injection margin answers the question

For a given voltage profile, by what factor can we change our power injections (uniformly at all buses) while still potentially having a solution?

Answer: $\eta = \left( \frac{V_0}{V_{\text{opt}}^{\text{slack}}} \right)^2$
Examples

• IEEE 14-Bus System

• IEEE 118-Bus System

• Tested many other systems and loadings
Controlled Voltage Margin

IEEE 14-Bus Continuation Trace: Bus 5

\[ \sigma \Big|_{\text{Inj mult}=1} = \frac{V_0}{V_{\text{slack}} \Big|_{\text{Inj mult}=1}^{\text{opt}}} \]

\[ = \frac{1.06}{0.5261} \]

\[ = 2.0148 \]

\[ V_{\text{slack}} = V_0 \]

\[ = 1.06 \]
Power Injection Margin

IEEE 14-Bus Continuation Trace: Bus 5

\[ \eta_{\text{Inj mult}=1} = \left( \frac{V_0}{V_{\text{slack}}^{\text{opt}}_{\text{Inj mult}=1}} \right)^2 \]

\[ = \left( \frac{1.06}{0.5261} \right)^2 \]

\[ = 4.0595 \]

\[ V_{\text{slack}} = V_0 \]

\[ = 1.06 \]

\[ V_{\text{slack}} = V_{\text{slack}}^{\text{opt}}_{\text{Inj mult}=1} \]

\[ = 0.5261 \]
## IEEE 14 Bus System

<table>
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<th>Injection Multiplier</th>
<th>Newton-Raphson Converged?</th>
<th>$V_0$</th>
<th>$V_{opt}^{slack}$</th>
<th>dim(null(A))</th>
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Voltage Margin

IEEE 14-Bus Voltage Margin $\sigma$ vs. Injection Multiplier

![Graph showing the relationship between IEEE 14-Bus Voltage Margin $\sigma$ and Injection Multiplier. The graph illustrates a decreasing curve as the Injection Multiplier increases.]
IEEE 14-Bus Continuation Trace: Bus 5

Voltage and Power Injection Margins

\[ \sigma_{|\text{Inj mult}=5} = \frac{V_0}{V_{\text{opt}}_{|\text{Inj mult}=5}} = \frac{1.06}{1.1764} = 0.9011 \]

\[ \eta_{|\text{Inj mult}=5} = \left( \frac{V_0}{V_{\text{opt}}_{|\text{Inj mult}=5}} \right)^2 = \left( \frac{1.06}{1.1764} \right)^2 = 0.8119 \]

\[ V_{\text{slack}} = V_{\text{opt}}_{|\text{Inj mult}=5} = \sqrt{5} V_{\text{opt}}_{|\text{Inj mult}=1} = 1.06 \]

\[ V_{\text{slack}} = V_0 = 1.1764 \]
IEEE 118-Bus Continuation Trace: Bus 44

\[ \sigma_{\text{Inj mult}=1} = \frac{V_0}{V_{\text{slack}}^{\text{opt}} \bigg|_{\text{Inj mult}=1}} \]

\[ = \frac{1.035}{0.5724} \]

\[ = 1.8082 \]

\[ V_{\text{slack}} = V_0 \]

\[ = 1.035 \]

\[ V_{\text{slack}} = V_{\text{slack}}^{\text{opt}} \bigg|_{\text{Inj mult}=1} \]

\[ = 0.5724 \]
Power Injection Margin

IEEE 118-Bus Continuation Trace: Bus 44

\[ \eta_{\text{Inj mult}=1} = \left( \frac{V_0}{V_{\text{slack}}^{\text{opt}}_{\text{Inj mult}=1}} \right)^2 \]
\[ = \left( \frac{1.035}{0.5724} \right)^2 \]
\[ = 3.2695 \]

\[ V_{\text{slack}} = V_0 \]
\[ = 1.035 \]

\[ V_{\text{slack}} = V_{\text{slack}}^{\text{opt}}_{\text{Inj mult}=1} \]
\[ = 0.5724 \]
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Voltage Margin
IEEE 118 Bus System
IEEE 118-Bus Continuation Trace: Bus 44

\[ \sigma_{\text{Inj mult}=4} = \frac{V_0}{V_{\text{opt}}_{\text{slack}}_{\text{Inj mult}=4}} \]
\[ = \frac{1.035}{1.1448} \]
\[ = 0.9041 \]

\[ \eta_{\text{Inj mult}=4} = \left( \frac{V_0}{V_{\text{opt}}_{\text{slack}}_{\text{Inj mult}=4}} \right)^2 \]
\[ = \left( \frac{1.035}{1.1448} \right)^2 \]
\[ = 0.8174 \]

\[ V_{\text{slack}} = V_0 \]
\[ = \sqrt{4} V_{\text{opt}}_{\text{slack}}_{\text{Inj mult}=1} \]
\[ = 1.1448 \]
Alternate Power Injection Profiles

Power injection margin is in the direction of a uniform, constant-power-factor injection profile

We can alternatively specify any profile that is a linear function of powers and squared voltages

- However, insolvability condition \( \eta < 1 \) is not necessarily valid

\[
\begin{align*}
\text{max } \eta & \quad \text{subject to} \\
\text{trace } (Y_k W) &= f_k \left( P, Q, V^2, \eta \right) \quad \forall k \in \{PQ, PV\} \\
\text{trace } (\tilde{Y}_k W) &= g_k \left( P, Q, V^2, \eta \right) \quad \forall k \in \{PQ\} \\
\text{trace } (M_k W) &= \alpha V_0 \quad \forall k \in \{PV\} \\
\text{trace } (M_{\text{slack}} W) &= V_0 \\
W &\succeq 0
\end{align*}
\]
Reactive Power Limits

Previous work models generators as ideal voltage sources.

Detailed models limit reactive outputs:
- Limit-induced bifurcations

Two approaches to modeling these limits:
- Mixed-integer semidefinite programming
- Infeasibility certificates using sum of squares programming
MISDP Formulation

Model reactive power limits using **binary variables**

\[
\begin{align*}
\text{max} & \hspace{0.5cm} \eta \\
\text{subject to} & \\
\text{tr} \left( Y_k W \right) &= P_k \eta \\
\text{tr} \left( \bar{Y}_k W \right) &= Q_{Dk} \eta \\
\left\{ \text{tr} \left( \bar{Y}_k W \right) \right\} &\geq Q_k^{\max} \psi_{Uk} + Q_k^{\min} (1 - \psi_{Uk}) \\
\left\{ \text{tr} \left( \bar{Y}_k W \right) \right\} &\leq Q_k^{\min} \psi_{Lk} + Q_k^{\max} (1 - \psi_{Lk}) \\
\left\{ \text{tr} \left( M_k W \right) \right\} &\geq (V_k^*)^2 (1 - \psi_{Uk}) \\
\left\{ \text{tr} \left( M_k W \right) \right\} &\leq (V_k^*)^2 (1 - \psi_{Lk}) + d \psi_{Lk} \\
\psi_{Lk} + \psi_{Uk} &\leq 1 \\
\sum_{k \in \{PV, S\}} (\psi_{Lk} + \psi_{Uk}) &\leq n_g - 1 \\
W &\geq 0 \\
\psi_{Uk} &\in \{0, 1\} \hspace{0.5cm} \psi_{Lk} \in \{0, 1\} \\
\text{for some large } d
\end{align*}
\]

Insolvability
MISDP Formulation

Model reactive power limits using binary variables

\[
\begin{align*}
\psi_U &= 0 \\
\psi_L &= 0
\end{align*}
\]
MISDP Formulation

Model reactive power limits using **binary variables**

\[
\begin{align*}
\psi_U &= 1 \\
\psi_L &= 0
\end{align*}
\]

Insolvability
Model reactive power limits using **binary variables**

\[
\begin{align*}
\max & \quad \eta \\
\text{subject to} & \quad \forall k \in \{PQ, PV\} \\
& \quad \forall k \in \{PV, S\} \\
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& \quad \forall k \in \{PV, S\} \\
\end{align*}
\]
Reactive Power Limits Results

14-Bus System Power vs. Voltage

<table>
<thead>
<tr>
<th>System</th>
<th>Trace Nose Point</th>
<th>$\eta^{max}$</th>
<th>Infeasibility Certificate</th>
</tr>
</thead>
<tbody>
<tr>
<td>14-bus</td>
<td>1.3522</td>
<td>1.3522</td>
<td>1.36</td>
</tr>
<tr>
<td>30-bus</td>
<td>2.8609</td>
<td>2.8609</td>
<td>2.86</td>
</tr>
<tr>
<td>57-bus</td>
<td>1.6486</td>
<td>1.6486</td>
<td>1.65</td>
</tr>
</tbody>
</table>
Conclusions

• Cast the problem of computing voltage stability margins as an optimization problem – to minimize the slack bus voltage.

• Calculated voltage stability margins – power injection/flows, and controlled voltages.

• Tested with numerical examples

Advantages:

• Eliminates repeated solution (multiple power flows, continuation power flows)

• Often offers provably globally optimal results

• Works when the margin is negative, i.e. when there isn’t a solution.
Related Publications


Questions?
Extra Slides: Feasibility
For lossless systems:

1. Show a solution exists for zero power injections at PQ buses and zero active power injection at PV buses, for $\alpha = 1$.

2. Use implicit function theorem to argue that perturbations to zero power injections solutions also exist. Specifically choose one in the direction of desired power injection profile.

3. Exploit the quadratic nature of power flow equations to scale voltages and power to match injection profile.
Feasibility: Zero Power Injection Solution

Easy: Construct a solution.

- Open Circuit PQ buses for zero power, zero current injection.
- Use Ward-type reduction to eliminate PQ buses. (not really necessary, but clean) 
- Choose uniform angle solution for all buses.
- Directly use reactive power flow equation to calculated the reactive power injections at generator buses.

\[
P_k = V_k \sum_{i=1}^{n} V_i (G_{ik} \cos(\delta_k - \delta_i) + B_{ik} \sin(\delta_k - \delta_i))
\]

\[
Q_k = V_k \sum_{i=1}^{n} V_i (G_{ik} \sin(\delta_k - \delta_i) - B_{ik} \cos(\delta_k - \delta_i))
\]
Note: Zero Power Injection Solutions for Lossy Systems

- Not all systems have a zero-power injection solution

\[ P_{PV} = gV_{PV}^2 - V_{PV}V_{slack} \left( g \cos(\theta) + b \sin(\theta) \right) \]

- Ability to choose \( \theta \) such that \( P_{PV} = 0 \) depends on
  - Ratio of \( V_{PV} \) to \( V_{slack} \)
  - Ratio of \( g \) to \( b \)

- Systems with small resistances and small voltage magnitude differences are expected to have a zero power injection solution
Nearby Solutions

A nearby non-zero solution exists

\[ f((V_{Di} + jV_{Qi}) + (\Delta V_{Di} + j\Delta V_{Qi})) = \Delta P_i + j\Delta Q_i \]

provided the Jacobian is nonsingular. For a connected lossless system at the zero-power injection solution, the appropriate Jacobian is nonsingular, generically.
Exploit the quadratic nature of power flow equations to scale voltages to match desired power profile:

\[ f(\beta(V_{Di} + jV_{Qi}) + \beta(\Delta V_{Di} + j\Delta V_{Qi})) = \beta^2(\Delta P_i + j\Delta Q_i) \]

\[ = P_i + jQ_i \]
Extra Slides: Infeasibility Certificates
Infeasibility Certificates

Guarantee that a system of polynomial is infeasible

\[ f_i(x) = 0 \quad i = 1, \ldots, m \]

\[ g_i(x) \geq 0 \quad i = 1, \ldots, p \]

Positivstellensatz Theorem

\[
\text{ideal} \left( f_1, \ldots, f_m \right) = \left\{ f \mid f = \sum_{i=1}^{m} t_i f_i, \quad t_i \in \mathbb{R}[x] \right\}
\]

\[
\text{cone} \left( g_1, \ldots, g_p \right) = \left\{ g \mid g = s_0 \sum_i g_i + \sum_{\{i,j\}} s_{ij} g_i g_j + \sum_{\{i,j,k\}} s_{ijk} g_i g_j g_k + \cdots \right\}
\]

If

\[
F(x) \in \text{ideal} \left( f_1, \ldots, f_m \right) \quad \text{such that} \quad F(x) + G(x) = -1
\]

\[
G(x) \in \text{cone} \left( g_1, \ldots, g_p \right)
\]

then the system of polynomials has no solution
Power Flow in Polynomial Form

Power Injection and Voltage Magnitude Polynomials

\[ P_i = f_{P_i}(V_d, V_q) = V_{di} \sum_{k=1}^{n} (G_{ik}V_{dk} - B_{ik}V_{qk}) + V_{qi} \sum_{k=1}^{n} (B_{ik}V_{dk} + G_{ik}V_{qk}) \]

\[ Q_i = f_{Q_i}(V_d, V_q) = V_{di} \sum_{k=1}^{n} (-B_{ik}V_{dk} - G_{ik}V_{qk}) + V_{qi} \sum_{k=1}^{n} (G_{ik}V_{dk} - B_{ik}V_{qk}) \]

\[ V_i^2 = f_{V_i}(V_d, V_q) = V_{di}^2 + V_{qi}^2 \]

Reactive Power Limit Polynomials

\[ f_{V_i} = \left((V_i^*)^2 - V_i^- + V_i^+\right) \]
\[ Q_i^{\text{max}} - f_{Q_i} = x_i \]
\[ V_i^- x = 0 \]
\[ V_i^+ (Q_i^{\text{max}} - Q_i^{\text{min}} - x) = 0 \]
\[ Q_i^{\text{max}} - Q_i^{\text{min}} - x \geq 0 \]
\[ V_i^+ \geq 0, \quad V_i^- \geq 0, \quad x_i \geq 0 \]
Find a sum of squares polynomial of the form

\[ H \left( V_d, V_q, x, V^+, V^- \right) = \tau V_{q,\text{slack}} + \sum_{i \in \{PV, PQ\}} \lambda_i (f_{Pi} - P_i) + \sum_{i \in \{PQ\}} \gamma_i (f_{Qi} - Q_i) \]

\[ + \sum_{i \in \{S, PV\}} \left\{ \psi_{1i} \left( (V_i^*)^2 - V_i^- + V_i^+ - f_{Vi} \right) + \psi_{2i} (Q_i^{max} - f_{Qi} - x_i) + \psi_{3i} V_i^- x \right\} \]

\[ + \psi_{4i} \left( Q_i^{max} - Q_i^{min} - x \right) V_i^+ + s_{1i} \left( Q_i^{max} - Q_i^{min} - x \right) + s_{2i} V_i^+ + s_{3i} V_i^- + s_{4i} x_i \]

such that

\[ (-H \left( V_d, V_q, x, V^+, V^- \right) - 1) \text{ is sum of squares} \]

by finding polynomials \( \tau, \lambda, \gamma, \phi_1, \phi_2, \phi_3, \phi_4 \)

and sum of squares polynomials \( s_1, s_2, s_3, s_4 \)

Then the power flow equations have no solution.