PSERC

Techniques for the Evaluation of Parametric Variation in Time-Step Simulations

Final Project Report

Power Systems Engineering Research Center

Empowering Minds to Engineer the Future Electric Energy System

Techniques for the Evaluation of Parametric Variation in Time-Step Simulations

Final Project Report

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Power Systems Engineering Research Center

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Executive Summary

Predictions of dynamic performance in power systems often require computationally intensive software simulations of vast numbers of scenarios, with dependence on large numbers of inputs and parameters that are uncertain. In this context, each simulation study "run" represents one sample of behavior for a particular set of conditions and parameters values across the time steps of the simulation. The overall goal is typically one of ensuring acceptable behavior over a wide range of parameter values and conditions, typically over some time period of interest. Traditional methods of sampling over uncertain parameters, such as Monte Carlo simulations, become prohibitively costly. The work in this project explored methods of probabilistic collocation (PCM) as a means to greatly reduce the number of input parameter possibilities to be sampled, while still producing good approximations of the behavior of output of interest. Examples of output quantities of interest would include load bus voltages, generator frequencies, power flows, or even the computed impedance 'seen" from a set of relay measurements, to determine if the relay enters its zone of operation.

For example, transmission planners may be interested in a dynamic voltage stability study in which voltage dip at several key buses under uncertainties in such quantities as in tap-changer delays and load parameter values. The Probabilistic Collocation Method would provide a rigorous algorithm for selecting a small number of specific parameter value combinations to study, and a means for combining these results to best estimate the behavior of the bus voltage magnitudes of interest. In a large scale power flow analysis problem where one wants to find the effect of a small number uncertain parameters on a bus voltage or line flow may require 100's of simulations in a Monte Carlo approach. The same problem can be modeled with just a handful of simulations using the Probabilistic Collocation Method. The method reduces the complexity by assuming a structured polynomial mapping between the uncertain input parameter(s) and the output variable of interest, and by identifying a good set of data points (i.e., simulation results) for robustly approximating the mapping. The strength of PCM lies in its propensity for selecting simulation (or collocation) points in the high probability region of the input parameter(s) distribution, and it is this feature that makes PCM a computationally efficient modeling

technique. Besides illustrating the strength PCM for power systems applications, we discuss two new results that further tailor it to the power systems application. In particular, the work here demonstrates the improvement of the PCM approximation using sensitivity information, and the computation of error bounds for PCM approximations. We also demonstrated that in many cases for which the original presentation of the problem contain a large number of uncertain parameters, variable reduction techniques can improve computational efficiency further without significantly sacrificing accuracy.

We tested the Probabilistic Collocation Method on a 14 Bus IEEE test system. We first modeled the input-output relationships precisely. Then, using PCM, we developed an efficient approximation far fewer input variables. Results obtained were very promising.

Future Directions

This report shows the versatility and potential for use probabilistic collocation methods. Refinement, extensions, testing on systems of realistic size, and applications development are the next steps.

- *Justification:* We have corroborated most of our results with analytical proofs, and demonstrated their application in illustrative power systems examples.
- *Optimization:* Our work includes first steps towards applying the Probabilistic Collocation Method in solving optimization problems relevant to power systems applications. While rudimentary, this appears a promising application, in which the efficiency of the PCM method will allow optimization algorithms to more broadly "search" the set of feasible solutions. Future work is needed for using the method to handle optimization problems in more computationally efficient ways.
- *Software Package:* Many of the computational examples presented in our work were generated in software developed as Mathematica™ modules, taking advantage of that package's great flexibility and convenience in manipulating nonlinear functional descriptions and system models. However, for greater efficiency, future work may benefit from a version of the method coded in a programming language such as C++.

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1. Introduction

1.1 Background and Problem Overview

Even when provided with a specific system topology and known parameter values describing individual elements, the dynamics of a large-scale electric power network require computationally-intensive time step simulation to predict the evolution of its outputs of interest (power flows, generator frequency variations, load bus voltage variations, etc.). When the parameters of the system are uncertain (e.g., imprecisely known load composition, or gains and time constants for exciters, power system stabilizers, and governor controls), it is necessary to simulate the system over many different parameter sets to adequate characterize the output(s). This project was concerned with the problem of intelligently choosing simulation points (e.g., sets of parameter values selected for simulations) so as to characterize the outputs of complex processes with minimal effort.

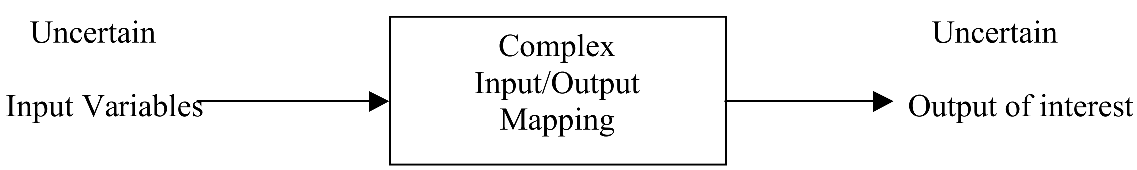


Figure 1-1: A general reduced order "black box" model representation.

We are specifically interested in characterizing the mapping between a set of uncertain parameters and an output of interest. We take the perspective that a low-order "black box" model can capture the mapping between the inputs and output. Such a "black box" model does not attempt to capture the operational intricacies of the system; rather, it tries to represent the relationship between the input variables and the output of interest based on observations of the system output at a finite set of input simulation points. Once the "black box" relationship between the input variables and the output of interest has been identified, such reduced order models could prove to be useful for analysis. However, to come up with input/output mapping or to characterize the output, the system has to be simulated for a set of input values.

"Economy" is the key word: techniques for coming up with such reduced order models with as few system simulations as possible could prove to be very useful. Traditionally, techniques such as brute force Monte Carlo simulation [16] were used for generating the mapping. The problem with such techniques is that they involve exhaustive simulations to characterize the outputs. If simulations are computationally intensive, characterization of outputs through exhaustive simulations may be infeasible.

Artificial Neural Networks (ANN) are also popular in the modeling arena; they are used for mimicking dynamic behavior of the system. Artificial Neural Networks map a set of input variables/patterns with corresponding output variables/patterns. A general ANN model consists of three layers viz. an input layer that carries the input information to the system, a hidden computational layer and an output layer. The input layer has connections, which has connection weights corresponding to it. The input values are multiplied by the weights and the weighted sum is formed. Each neuron has a threshold value (called bias) associated with it which is subtracted from the weighted sum. The computational layer applies an activation function to this weighted sum to produce the output. To determine the weights and biases and optimization procedure (called training) is used. We request the readers to refer to [20] and [21] for more information on ANN based modeling.

An alternative approach for intelligently choosing simulation points is to exploit probabilistic descriptions of the uncertain parameters. In other words, we would like to choose simulation points in such a way that the mapping between the parameter and output is accurately identified over the range of likely parameter values. The Probabilistic Collocation Method [1], [2], [3], [5] is a technique that can be used to model the deterministic relationship between uncertain parameters and an output of interest using polynomial functions. This is the approach that we shall take in this report.

The Probabilistic Collocation Method (PCM), also known as Deterministic Equivalent Modeling Method (DEMM), is a modeling technique that employs Gaussian quadrature [8] to characterize the relationship between uncertain input parameters and an output of interest. The output of interest is modeled as a polynomial of the uncertain input parameter(s). PCM was first used for global climate change studies [5]. In [1], [2] and [3] the authors apply PCM for modeling uncertainties in electric power systems. When probabilistic descriptions for the uncertain parameters are well known, it has been claimed that the Probabilistic Collocation Method (PCM) is more efficient compared to simulation techniques like Monte Carlo in terms of number of simulations required to capture the input-output relationship. For instance, a simple power systems load flow analysis problem where we are want to find the effect of a particular uncertain parameter on the bus voltage or line flow may require 100's or sometimes even 1000's of simulations in the Monte Carlo approach whereas the same problem can be modeled with just a handful of simulations using PCM. PCM reduces the complexity by assuming a structured polynomial mapping between the uncertain input parameter(s) and the output of interest and identifying a good set of simulations for correctly and robustly determining the mapping. The point selection is done based on Gaussian quadrature, which forms the crux of the theory behind PCM. Another interesting feature of PCM is that the same set of simulation points can be used for analyzing multiple output parameters.

It is interesting to consider some other modeling techniques for efficiently choosing simulation points under parameter uncertainty. The Stochastic Response Surface Method (SRSM) [14], [15] is an uncertainty modeling technique used mainly in the field of chemical and bio-medical engineering. In SRSM the inputs of the system are represented as functions of certain standard random variables (srvs) and each output under examination is expressed as a series expansion in terms of the srvs as multidimensional Hermite polynomials. The reasoning behind this representation is that it offers consistency, as the srvs are well behaved and mathematically tractable. The mapping between the input and the output can be established by estimating the coefficients of the output series expansion and this is achieved by collocation methods, like PCM, or regression methods.

The authors discuss PCM for this purpose and, notably, they discount its usefulness on the grounds that PCM becomes unwieldy when the number of input parameters is large. We have proposed a techniques to address this issue, which is one of our major contributions in this project. The authors adopt a regression-based collocation method for estimating the coefficients which they address as regression based SRSM. It requires twice as many collocation points as there unknown coefficients for estimation. Moreover, in [15], the authors claim that SRSM maybe may be more useful in the case of complex nonlinear models.

The Stochastic Collocation Method (SCM) [17] used mainly in the field of fluid dynamics, transforms the physical random variables to an artificial stochastic space with known properties, and then uses a collocation-based approach for modeling the relationship between the physical random variable and the output of interest.

Unlike PCM, the techniques mentioned above are quite complex to implement. PCM is appealing because it is simple and yet allows the evaluation of complicated output functions.

1.2 Report Organization

This report is organized into six chapters. Following the introduction of this Chapter 1, Chapter 2 presents the general theoretical background on the probabilistic collocation method, introducing the one-dimensional PCM, and discussing its underlying theory of Gaussian quadrature and orthogonal polynomials. It illustrates one-dimensional PCM with the help of general technical application examples. The succeeding Chapter 3 provides a generalization of PCM to handle multiple-correlated uncertain parameters as well as proposing power system specific ways to employ PCM with what is known as "boundary" load flow. Chapter 4 introduces optimization problems of interest in power systems that may be approached via PCM methods.

Information theoretic approaches for reducing the number of input uncertain variables is discussed in Chapter 5 and demonstrated through illustrative examples. In Chapter 6, we apply PCM to model the input-output relationship for a 14 Bus IEEE test system. Chapter 7 provides conclusions and future research directions.

2. Theoretical Background for the Probabilistic Collocation Method

2.1 General background

The **Probabilistic Collocation Method** is a means for developing a parametric model for the deterministic mapping between a stochastic input and an output (Figure 2-1), using only a small number of simulations of the system. In particular, ***n*th-order PCM** seeks to represent the mapping using an *n*th-degree polynomial whose coefficients are found by matching the model predictions with simulation outputs for a particular set of $n+1$ input values. The $n+1$ input values---henceforth called the **PCM points**---are specially chosen, in a manner that makes the fit robust to some possible errors in the model's parameterization. Specifically, the $n+1$ PCM points are chosen so that the mean output predicted by the model is identical to the actual mean output, if in fact the mapping is a polynomial of any degree less than or equal to $2n+1$. Thus, PCM specifies a low-order mapping that approximates a much higher-order (in other words, more detailed) mapping, in the sense that the mean output predicted by both mappings is identical.

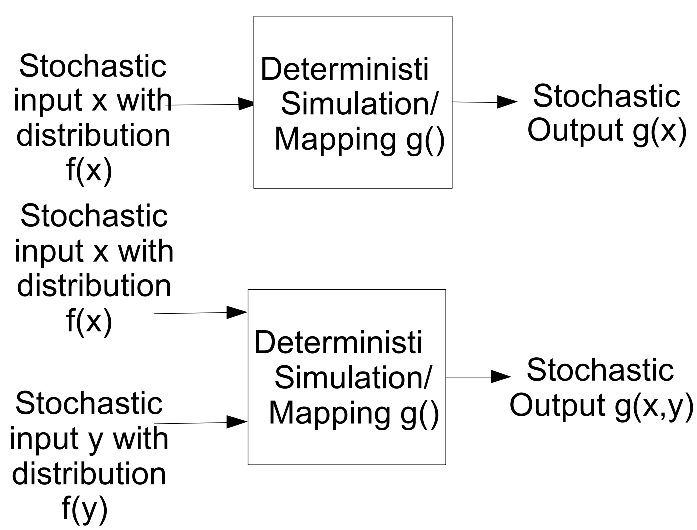


Figure 2-1: Mappings with single and multiple stochastic inputs.

PCM can be used to characterize the mapping g() and the probability distribution of the output with a small number of simulations.

The specialty of PCM lies in its propensity for selecting simulation (or collocation) points in the high probability region of the input distribution, and it is this feature that makes PCM a cost effective modeling technique. The theory behind PCM is based on the concepts of Orthogonal polynomials and Gaussian quadrature. So, before elucidating the PCM mechanism, we find it necessary to throw some light on the above mentioned concepts.

2.2 Orthogonal Polynomials and Gaussian Quadrature

Gaussian quadrature is a particular numerical integration technique. For our application here, Gaussian quadrature provides advantages over traditional numerical integration in the fact that it offers the freedom to select points at which the given function can be evaluated. Suitably exploiting this flexibility in selection of evaluation points allows the number of points at which the function has to be evaluated to be significantly reduced. Apart from the economy aspect, it has been claimed that the results obtained using Gaussian quadrature are more accurate compared to traditional numerical integration techniques like the Simpson's rule or the trapezoidal rule.

Gaussian quadrature uses orthogonal polynomials for the purpose of selecting points. The typical form of integrals in Gaussian quadrature is

$$\int_F f(x)g(x)dx \tag{2.1}$$

where $g(x)$, is an orthogonal polynomial, $f(x)$ is a non-negative weighing function defined in the connected space F, and the above expression defines an inner product.

Before going further into the theory behind Gaussian quadrature and the Probabilistic Collocation Method, it is useful to review some details concerning orthogonal polynomials.

Orthogonal Polynomials: As the name suggests, orthogonal polynomials are polynomials that are orthogonal to each other, specifically with respect to an inner product operation. We find it worthwhile to reproduce the definition of inner product and orthogonal polynomials from [2] and [8].

Given a real linear space of functions F, an inner product $\langle f,g\rangle$ (we represent inner product by angled brackets) defined on F is a function of $f,g\in F$ satisfying the following conditions:

$$\langle f+g,h\rangle = \langle f,h\rangle + \langle g,h\rangle \tag{2.2}$$

$$\langle \alpha f,g\rangle = \alpha\langle f,g\rangle = \langle f,\alpha g\rangle \text{, where } \alpha \text{ is a scalar} \tag{2.3}$$

$$\langle f,g\rangle = \langle g,f\rangle \tag{2.4}$$

$$\langle f,f\rangle > 0 \text{, if } f \neq 0 \tag{2.5}$$

For example, consider two polynomials $g(x)$ and $h(x)$ if $f(x)$ is any non-negative weighting function defined on the space, then $\langle g(x),h(x)\rangle \overset{\Delta}{=} \int_F f(x)g(x)h(x)dx$ is an inner product.

This expression is very important as it is the peculiar inner product that forms the basis for Gaussian quadrature integration and the Probabilistic Collocation Method. The polynomials $g(x)$ and $h(x)$ are said to be orthogonal if their inner product is zero.

Orthonormal polynomials

A set of polynomials in the space H are said to be orthonormal if and only if the following relationship exists for all $h_i(x)$ in H.

$$\langle h_i,h_j\rangle = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \tag{2.6}$$

The subscript of the polynomial indicates its degree; i.e., $h_i(x)$ has degree i. An important property of these orthonormal polynomials is that they are unique and they form a basis for the space for all polynomials. Another important property of these orthonormal polynomials is that their roots depend only upon the weighting function $f(x)$. Further, all the roots are contained in the space F, and each orthonormal polynomial h_i has exactly i roots. The roots of these polynomials form the collocation points for the Probabilistic Collocation Method. The set $\{h_i\}$ of orthonormal polynomials of increasing degree form the backbone of PCM.

Gaussian Quadrature

As mentioned before, Gaussian quadrature is a special numerical integration technique for integrals of the form

$$\int_F f(x)g(x)dx$$

In a nutshell, Gaussian quadrature seeks to obtain the best numerical estimate for the above integral and it does so by picking certain x values, evaluating $g(x)$ at these points, and the computing the integral. The x values are the roots of the orthogonal polynomials discussed in the previous section.

The result of Gaussian quadrature integration is the following formula:

$$\int_F f(x)g(x)dx \approx \sum_{i=1}^{n} f_i g(x_i) \tag{2.7}$$

The coefficients f_i depend on the weighting function and the function $g(x)$ is evaluated at abscissa values that are the roots of the nth orthogonal/orthonormal polynomial calculated with respect to the weighting function $f(x)$.

The above integral is exactly correct when $g(x)$ is a polynomial of degree (2n-1). Interestingly, the integral can be estimated using just n samples. This shows that the Gaussian quadrature has the ability to represent a higher order relationship using a lower order polynomial; PCM inherits this property from Gaussian quadrature.

<u>Proof for Gaussian Quadrature</u>

The polynomials in H up to and including order i form an orthonormal basis for the space of all polynomials of degree less than or equal to i. Then, a polynomial of order $(2n-1)$ can be expressed as follows:

$$g(x) = h_n(x)(a_{n-1}h_{n-1}(x) + \cdots + a_0 h_0(x)) + b_{n-1}h_{n-1}(x) + \cdots + b_0 h_0(x) \tag{2.8}$$

Note that $h_0(x)$ is a constant. Hence, by orthogonality, the Gaussian quadrature integral can be expressed as follows:

$$\int_F f(x)g(x)dx = b_0 \int_F f(x)h_0(x)dx \tag{2.9}$$

By evaluating the function $g(x)$ at the n roots of the orthogonal polynomial $h_n(x)$ we get the following set of linear equations:

$$\begin{bmatrix} g(x_1) \\ \vdots \\ g(x_n) \end{bmatrix} = \begin{bmatrix} h_{n-1}(x_1)\cdots h_0(x_1) \\ \vdots \qquad\qquad \vdots \\ h_{n-1}(x_n)\cdots h_0(x_n) \end{bmatrix} \begin{bmatrix} b_{n-1} \\ \vdots \\ b_0 \end{bmatrix} \tag{2.10}$$

To solve for b_0, we need to invert the above expression. If $h_0(x)$ is chosen to be equal to 1, as it generally is then our desired result is b_0.

$$\begin{bmatrix} b_{n-1} \\ \vdots \\ b_0 \end{bmatrix} = \begin{bmatrix} h_{n-1}(x_1)\cdots h_0(x_1) \\ \vdots \qquad\qquad \vdots \\ h_{n-1}(x_n)\cdots h_0(x_n) \end{bmatrix}^{-1} \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_n) \end{bmatrix} \tag{2.11}$$

$$b_0 = \int_F f(x)g(x)dx \approx \sum_{i=1}^{n} f_i g(x_i) \tag{2.12}$$

Where, the weights f_i are given by the last row of the matrix in (2.11).

2.3 One Dimensional PCM

Given an input random variable x with probability density function (PDF) $f(x)$ and an output of interest, we seek to approximate the functional mapping *g(x)* that transforms the input to the output. Notice that the mean value of the output in this case is given by

$$\mathrm{E(x)} = \int_F f(x)g(x)dx \tag{2.13}$$

Gaussian quadrature allows us to choose *n*+1 points such that, for any *g*(x)* that is a polynomial of order less than or equal to *2n+1*, and for which the integral is the same. Thus, the mean value predicted by the degree-*(n+1)* polynomial passing through these points is the same as the mean predicted by any polynomial of degree less than or equal to *2n+1* that passes through the points. Equivalently, the degree *(n+1)* polynomial suffices to capture the mean output, if the mapping is indeed a polynomial of degree less than or equal to *2n+1*. The Gaussian quadrature points (in this case the PCM points) are determined by computing the first *n+1* **orthogonal polynomials** with respect to *f(x).*

Once the (n+1) PCM points are generated, the function under study is simulated at these points. The n^th order PCM polynomial will be of the form

$$\hat{g}(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{2.14}$$

By substituting the (n+1) PCM points in the above equation we get n equations, and by solving these equations using the value of the function under study at these (n+1) PCM points, we can obtain the coefficients of the n^th order PCM polynomial.

A General Scientific Example

To illustrate a classic application of Probabilistic Collocation Method, we first examine an example from the field of physical chemistry.

The "Ideal Gas" law [22] is an equation that describes the physical behavior of an ideal gas. It combines three primitive gas laws viz. Boyle's Law, Charles's Law and Avogadro's Law. The equation relates the pressure P, volume V, and the temperature T of an ideal gas. In the same vein, an ideal gas is one whose physical properties satisfy the ideal gas equation.

The ideal gas law is stated as follows

$$PV = nRT \tag{2.15}$$

P is the pressure of the gas.
V is the volume of the gas.
n is the number of moles of the gas.
R is the universal gas constant, R = 0.0821.
T is the temperature in Kelvin.

For our purpose this brief introduction to the "Ideal Gas" law would suffice. Typical problems related to the "Ideal Gas" equation would be finding the value of one of the entities given the rest.

Our first PCM example in this report is an attempt to model the "Ideal Gas" law. Having the actual relationship in hand helps as the PCM generated polynomial model can be compared with the actual analytical relationship. Another reason for the choice of this example is just to illustrate the prospect of PCM as an algorithm that can be used in several fields of study.

For a particular gas, we have made the following assumptions, with the temperature T at absolute zero (273 K) we want to find the volume occupied by one mole of the gas in liters (1 liter = 0.264172051 gallon) when the pressure is randomly distributed between 0.6 and one atmosphere (atm).

We would like to remind the readers that this example is only for the purpose of illustrating PCM. Otherwise, PCM, or any other uncertainty analysis technique for that matter, would be obviated for such an example because the relationship between the uncertain parameter and the output of interest can trivially be computed analytically.

To make the analysis interesting and, more practically, to show that PCM can handle any kind of distribution, we have chosen an unconventional probability density function (PDF) for the pressure.

$$f(P) = \begin{cases} 4P - (2/5), 0.6 < P < 0.8 \\ (4P - 1), 0.8 < P < 1 \end{cases} \tag{2.16}$$

The PDF is depicted below in Figure 2-2.

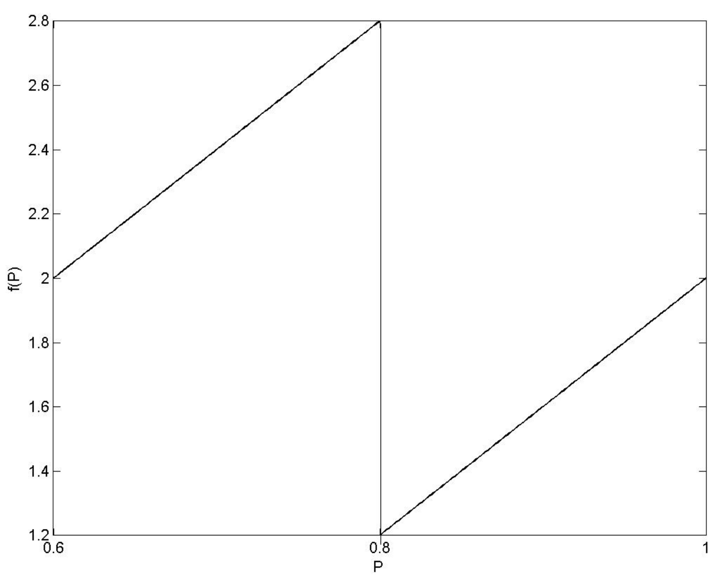


Figure 2-2: Plot of the distribution for the pressure, P.

Using the above PDF the first few orthogonal polynomials are generated; the roots are the pressure values for which the volume is calculated using the Ideal Gas equation. The table below lists the orthogonal polynomials up to order 4 with their roots.

Table 2-1: Orthogonal polynomials and their roots.

Orthogonal Polynomials
$h_0(P) = 1$
$h_1(P) = 8.69P - 7.03$ $P = 0.8933$
$h_2(P) = -84.34P^2 - 135.89P + 53.61$ $P = \{0.6905, 0.9207\}$
$h_3(P) = 829.75P^3 - 1997.69P^2 + 1583.37P - 412.96$ $P = \{0.6481, 0.8022, 0.9573\}$
$h_4(P) = -8240.93P^4 - 26450P^3 + 31553P^2 - 16576.4P + 3235.28$ $P = \{0.6293, 0.7354, 0.8714, 0.9734\}$

The volume is then calculated by substituting the roots of the orthogonal polynomials into the gas equation and the coefficients of the PCM polynomial are obtained by solving. For example, if we want the PCM quadratic polynomial, say g(P), for the relation under study, we have to use the roots of the 3rd order orthogonal polynomial. In general, the roots of the order n orthogonal polynomial are used to generate the order (n-1) PCM polynomial.

The roots of the 3rd order orthogonal polynomial are 0.648104, 0.802175 and 0.957301. The corresponding volume values are

$$V_1 = 34.5661$$

$$V_2 = 27.9271$$

$$V_3 = 23.4016$$

We need a polynomial of the form $aP^2 + bP + c$. Using the values of the roots and the corresponding g() values, we obtain three equations which can be solved to give us the values of the coefficients.

Thus, the PCM quadratic for the relationship is as follows:

$$g(P) = 45.01P^2 - 108.37P + 85.894 \tag{2.12}$$

We would also like to present the linear and cubic PCM approximations:

$$g(P) = -35.24P + 56.78 \tag{2.13}$$

$$g(P) = -57.048P^3 + 183.11P^2 - 218.45P + 114.77 \tag{2.14}$$

Analysis

We present plots comparing the polynomials generated by PCM with the actual function. The actual function plot is obtained by exhaustively simulating the equation $V = \dfrac{nRT}{P}$. The plots show the accuracy of PCM. Figure 2-4 reveals that the PCM quadratic is very close to the actual function. From Figure 2-5 it can be observed that the PCM cubic is so close to the actual function that it is hard to differentiate between them. The power of PCM is such that with just four simulations we are able to model the relationship between the volume and pressure. An important attribute of PCM, as mentioned before, is its characteristic of identifying the mean value of the output correctly. To illustrate this, Table 2-2 presents the expected value and variance of the PCM polynomials along with those of the actual function.

Table 2-2: Comparison of mean and variance values of different order PCM predictions

Function	Expected Value and Variance
PCM 1^{st} Order Polynomial	E = 28.2569, σ^2 = 16.4515
PCM 2^{nd} Order Polynomial	E = 28.2668, σ^2 = 17.3091
PCM 3^{rd} Order Polynomial	E = 28.267, σ^2 = 17.333
Actual Function	E = 28.2807, σ^2 = 17.35

PLOTS:

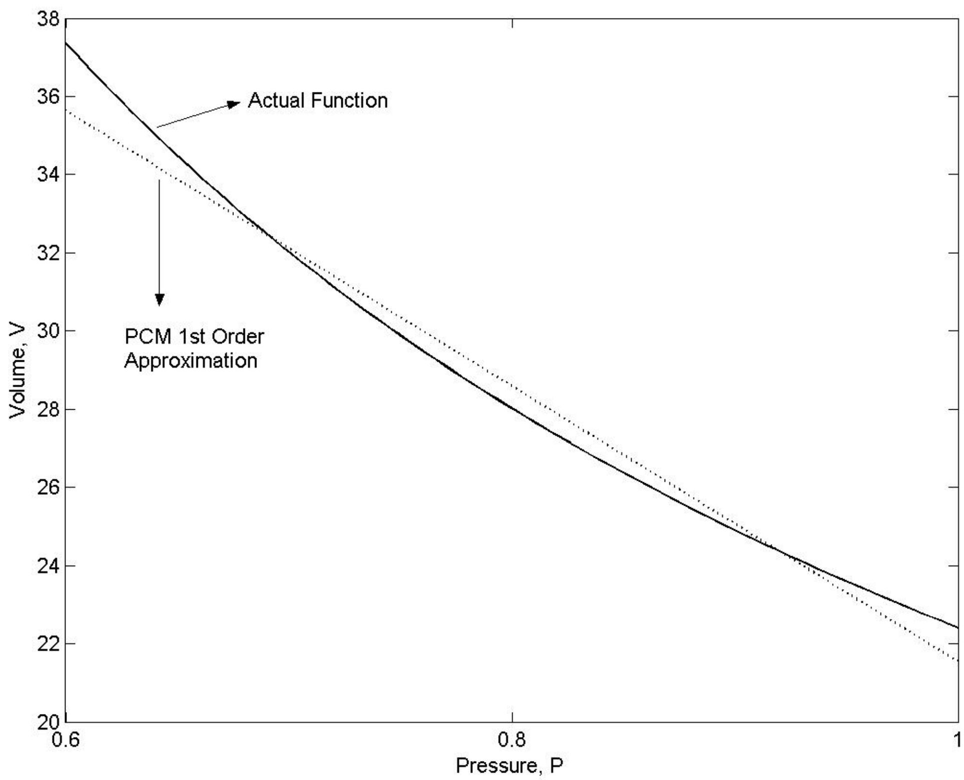


Figure 2-3: PCM linear approximation and the actual function plotted through exhaustive simulation. The solid line represents the actual function and the dotted line is the PCM linear polynomial.

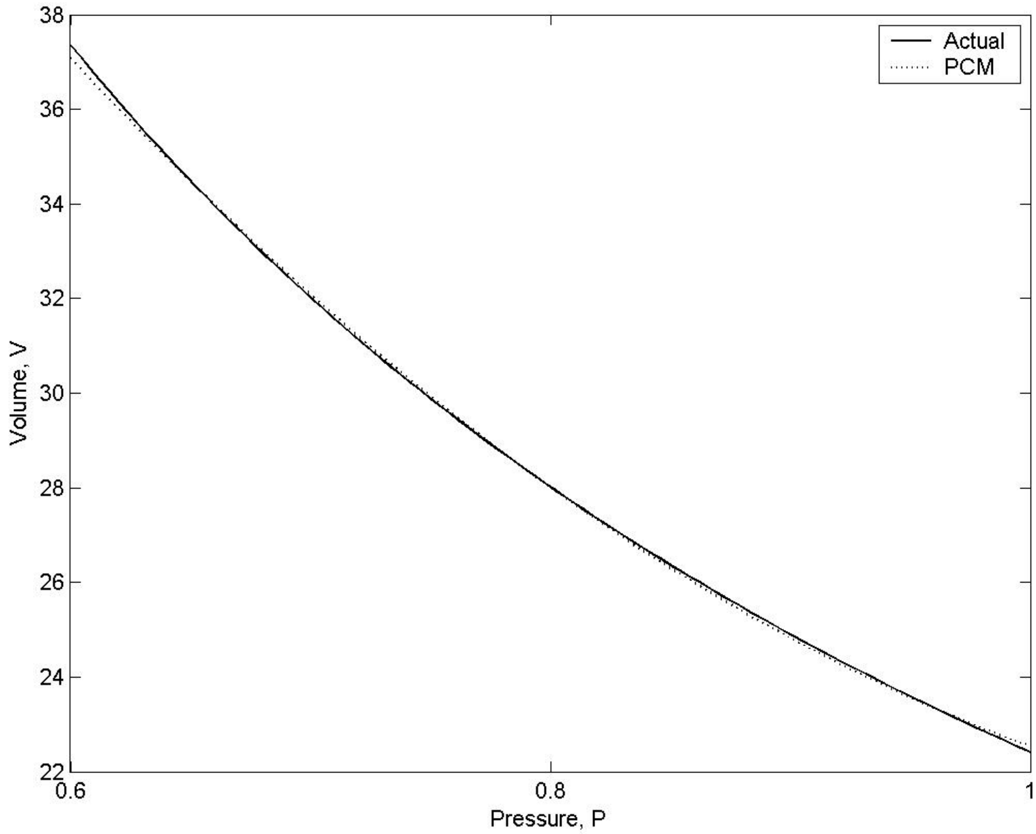


Figure 2-4: PCM 2nd Order and the actual mapping. The solid line represents the actual mapping and the dotted line is the quadratic PCM. The two plots are identical except at the upper endpoint where they disagree slightly.

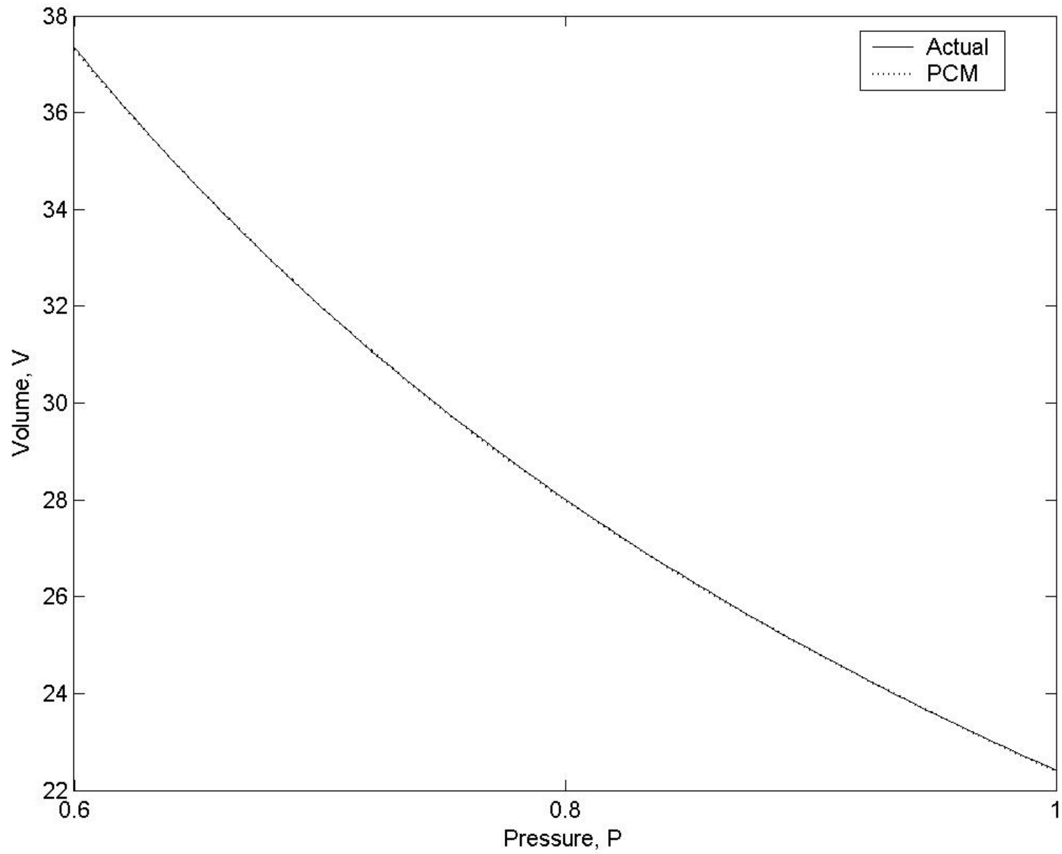


Figure 2-5: 3rd Order PCM and the actual mapping. The solid line represents the actual mapping and the dotted line is the cubic PCM polynomial.

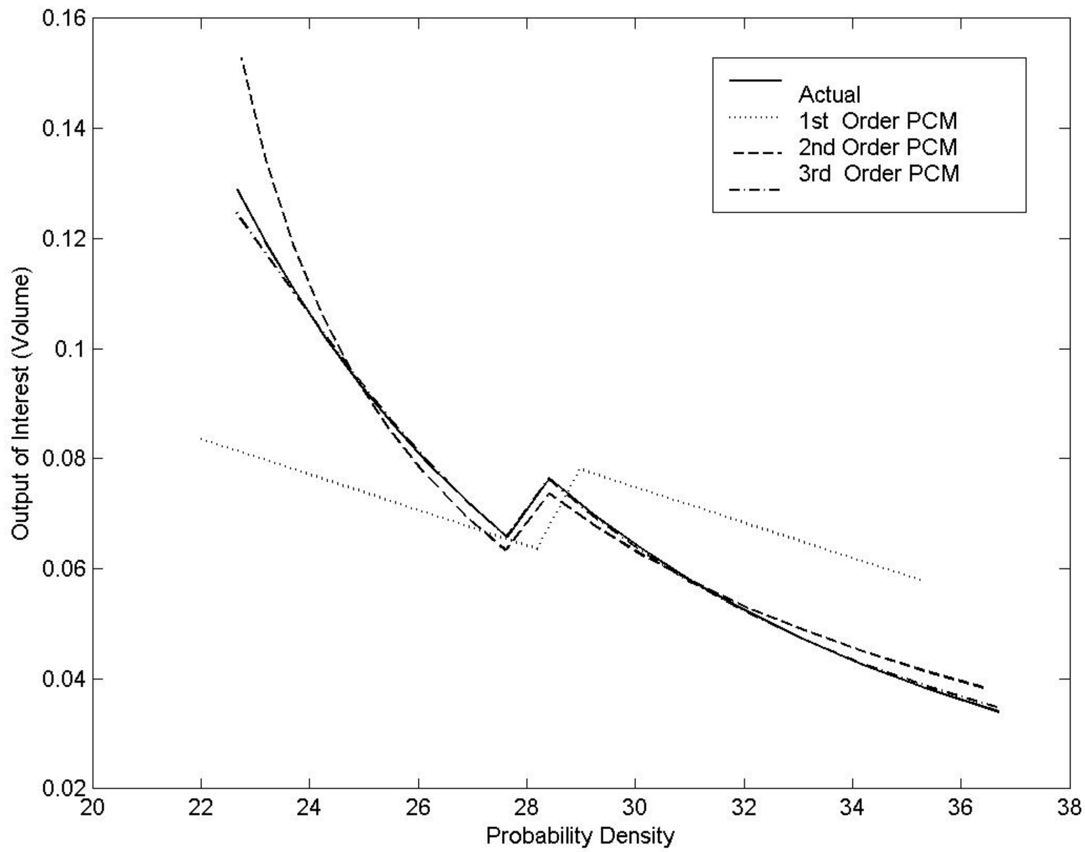


Figure 2-6: Comparison of output probability distributions. Solid line – actual function's distribution. The dotted, dash, and dot-dash – distributions based on the PCM 1st, 2nd and 3rd order approximations.

2.4 Improving PCM

In the remainder of this chapter, we present two new results concerning PCM, namely the improvement of the PCM approximation using sensitivity information and the computation of error bounds for PCM approximations. Theses analyses follow naturally from well-known results in the Gaussian quadrature community, but have not heretofore been considered in the study of PCM. In presenting these results, we hope to briefly introduce the reader to relevant literature on quadrature, and to highlight some distinctions in analysis goals for quadrature and PCM, respectively.

Some simulation programs not only can compute the output for a particular parameter value, but also can determine the sensitivity of the output to the parameter. For instance, efficient means for characterizing sensitivities of power flow outputs to loading parameters have been developed, and when sensitivity information is available, we might expect to obtain a more detailed characterization of the input-output mapping. That is, since we have additional information of the input-output mapping (in particular, knowledge of the derivative of this mapping at the simulation points), we should be able to generate a more accurate approximation of the mapping.

Let us assume that the output $g(x)$ and the sensitivity of the output $\frac{d(g(x))}{dx}$ to the parameter x have been found at the $(n+1)$ PCM points $x_1^*,\cdots,x_{n+1}^*$. Then we recommend fitting the mapping using a degree-$(2n+1)$ polynomial (as opposed to a degree n polynomial for PCM), which matches both the output data and output sensitivities. That is, we recommend approximating the mapping using the degree-$(2n+1)$ $\hat{g}_d(x)=\alpha_1 x^{2n+1}+\cdots\alpha_{2n+1}x+\alpha_{2n+2}$ that satisfies the equality:

$$\hat{g}_d(x_i^*)=g(x_i^*), 1\le i\le n+1$$

$$\frac{d\hat{g}_d(x_i^*)}{dx}=\frac{dg(x_i^*)}{dx}, 1\le i\le n+1$$

It is easy to check that these $(2n+2)$ equalities give $(2n+2)$ independent linear relations for the parameters $\alpha_1,\cdots\alpha_{2n+2}$, and, hence, that $\hat{g}_d(x)$ is determined uniquely from the known outputs and sensitivities. Let us refer to the approximation $\hat{g}_d(x)$ as the n^{th} order <u>PCM-with-sensitivity approximation.</u>

It is worth making several observations about the PCM-with sensitivities approximation. First, we note that if the actual mapping $g(x)$ is a degree $(2n+1)$ polynomial, the n^{th} PCM-with-sensitivity approximation is identical to the actual mapping. We should not be surprised that the mapping can be identified exactly, since we have available $(2n+2)$ independent data points ($n+1$ output values and $n+1$ sensitivities). In fact, any set of $(2n+2)$ independent measurements can be used to identify the mapping and the PCM-with-sensitivities approach is only special in that only $(n+1)$ simulations may be needed (if sensitivities are automatically generated).

What is more surprising is that the PCM-with-sensitivities approximation $\hat{g}_d(x)$ can be guaranteed to be close to the actual mapping even when the mapping is not a polynomial of degree *2n+1*. It turns out that $\hat{g}_d(x)$, known in the literature as a *Hermite polynomial*, is used in generating error bounds for Gaussian quadrature ([25], see also [26] for a succinct description of Markov analysis).

2.5 Chapter Summary

The results illustrate the following:

- The accuracy of PCM in modeling a deterministic mapping between uncertain parameter and an output of interest
- The economy of PCM.

In the chapters to follow, we will delve deeper into these two results.

3. Multiple Correlated Inputs: Conditional PCM

3.1 Background

Our studies of one-dimensional PCM suggest that it is very economic computationally as compared to techniques like Monte Carlo simulation. In this chapter, we present an extension of PCM for handling systems with multiple, correlated uncertainties.

For convenience, let us first discuss our generalization of PCM to systems with *two* correlated, uncertain inputs (see Figure 2-1). We call this generalization **two-dimensional PCM**. We assume that the two uncertain inputs x and y are jointly distributed according to a density function $f(x,y)$ that is non-zero over a finite, convex two-dimensional domain A. Our aim is to identify the mapping $g(x,y)$ that specifies the output in terms of these inputs. We assume that this mapping can be approximated by a two-dimensional multinomial of the form:

$$g^*(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij} x^i y^j \tag{3.1}$$

Henceforth, we refer to $g^*(x,y)$ as a **generalized polynomial of degree *n***. We feel that a generalized polynomial representation for a two-dimensional mapping is appropriate because (as in the one-dimensional case) higher-degree generalized polynomials provide more and more detailed representations of the mapping. More specifically, an order n generalized polynomial representation allows us to specify a set of polynomial mappings between each single input and the output, given the other input. To determine the coefficients in (3.1), we simulate the output for a particular set of $(n+1)^2$ input pairs, which we again call PCM points. From the corresponding $(n+1)^2$ outputs at the PCM points, we determine the coefficients at low computational cost by simply solving a system of linear equations. As in the one-dimensional case, the success of two-dimensional PCM depends strongly on appropriate choice of the PCM points.

3.2 The Two-Dimensional PCM Algorithm

We propose the following algorithm for choosing the PCM points:

1. We compute the marginal distribution for the input x as $f(x) = \int_A f(x, y)dy$. We then find the degree-($n+1$) orthogonal polynomial with respect to $f(x)$, and find the roots of this polynomial. Notice that these are the x values that we would choose as PCM points if we were applying one-dimensional PCM of order n to find a mapping between x and an output. Let us label these points $x_1, \dots x_{n+1}$.

2. We compute the conditional distributions $f(y \mid x_i) = \dfrac{f(x_i, y)}{f(x_i)}$. We then find the degree ($n+1$) orthogonal polynomials with respect to each distribution, and find the roots of these polynomials. Let us call the roots of the orthogonal polynomial with respect to $f(y \mid x_i)$ as $y_1(x_i), \dots y_{n+1}(x_i)$.

3. We use the $(n+1)^2$ pairs of inputs $[x_i, y_j(x_i)]$, $1 \le i \le n+1, 1 \le j \le n+1,$ as the PCM points.

The following analytical results (presented without proof) can be deduced for two-dimensional PCM; these results motivate use of the method:

1. Given that the input x is any one of the values $x_1, \dots x_{n+1}$, the mean output is correctly predicted by two-dimensional PCM whenever the actual mapping is a generalized polynomial of degree less than or equal to $2n+1$. Also, from continuity arguments, we can argue that the mean output predicted by PCM is nearly correct for inputs x that are close to one of the points $x_1, \dots x_{n+1}$. Since the points $x_1, \dots x_{n+1}$ are chosen to reflect the high-probability domain for the input x (this is one of the benefits of one-dimensional PCM), two-dimensional PCM predicts the mean output correctly given likely values for x.

2. In the special case that x and y are in fact independent, the (unconditioned) mean value for the output is correctly predicted by PCM whenever the mapping is a generalized polynomial of degree less than or equal to $2n+1$. Further, in the more general case that x and y are not independent but the rth-conditional moment for y given x is an rth-order polynomial, PCM predicts the output mean whenever the actual mapping is a true two-dimensional polynomial of degree less than or equal to $2n+1$ (i.e., a sum of monomial terms, each of which has total degree less than or equal to $2n+1$).
3. The PCM points always fall within the region A, so that we should be able to simulate a meaningful output for each PCM point.

We note that PCM can easily be generalized to identify mappings between three or more uncertain inputs and an output. As in two-dimensional PCM, we can select PCM points for higher-dimensional PCM recursively from a sequence of marginal and conditional distributions. These higher-dimensional PCM algorithms are amenable to the same analyses as two-dimensional PCM.

In the remainder of this chapter, we apply two-dimensional PCM to characterize the mapping between the two uncertain loads and the load flow voltage at a bus in a power system. Our study is in the context of a toy example obtained from [10], and is not meant to provide a comprehensive depiction of load flow uncertainties by any means. Our primary purpose is to illustrate two-dimensional PCM, and to explore some potential benefits and caveats of using PCM to characterize load flow solutions.

PCM-based characterization of load flow voltages falls within the broad class of Probabilistic Load Flow (PLF) algorithms (see [10] for a summary of some work on PLF) These are methods for computing uncertainties on load flow solution parameters (e.g., bus voltages or line loadings), given uncertainty distributions on load powers and other system parameters. A full study of the literature on PLF is beyond the scope of this report, but we present a few general concepts. As discussed in the literature (e.g., [7] and [11]), PLF algorithms are either based on Monte Carlo simulation techniques, on exact analysis, or on some combination of these. Very often, analytical methods assume a load flow model that is linearized around one or multiple equilibria, and require some structure (e.g., Gaussian) in the parameter distributions. Monte Carlo techniques account for the nonlinearities in the load flow solution and allow for general input parameter distributions, but are computationally intensive. As an alternative to PLF, load flows for systems with uncertain inputs have also been characterized by identifying limits on the output variables given limits or distributions on the inputs (e.g., [7], [12]). These methods, called boundary load flow algorithms, have recently been combined with techniques that provide fuzzy-set descriptions of output variables, given fuzzy descriptions of input variables [7].

3.3 Power System Examples

We believe that PCM can contribute to PLF analysis by providing an intelligent simulation strategy and also by providing a method for meshing probabilistic and boundary methods.

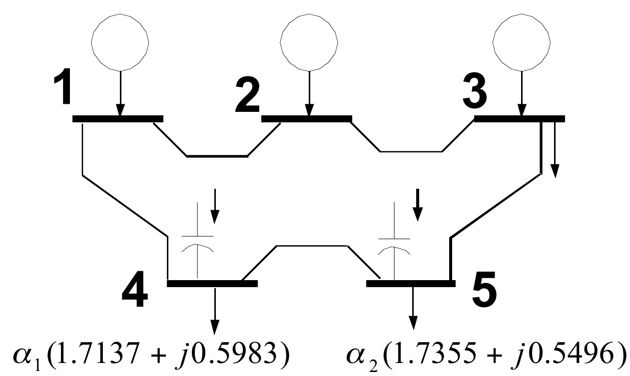


Figure 3-1: Load flow example. PCM applied to characterize the voltage at bus *4*, given that the loads at buses *4* and *5* are uncertain.

We apply PCM to find the PLF solution in the small power system example shown in Figure 3-1. In this example, we assume that the scaling parameters (inputs) x and y (see equation 3.1) which specify the load power magnitudes at buses *4* and *5*, are jointly distributed as shown in Figure 3-2. The positive correlation is meant to reflect that load requirements tend to be correlated with external parameters (e.g., temperature) which are roughly constant over a set of loads. Our output variable is the magnitude of the voltage at bus *4*. Application of PCM to this example first requires computation of the PCM points; the nine points for second-order PCM are shown on Figure 3-2. Using the PCM collocation points, we characterize the mapping between the inputs and output. The second-order generalized representation for the mapping found using PCM is the following:

$$g^x(x,y) = -0.041x^2y^2 + 0.2x^2y - 0.24x^2 + 0.12xy^2 - 0.61xy + 0.72x - 0.1y^2 + 0.45y + 0.48$$

This predicted mapping is compared to the actual mapping (generated through exhaustive simulation) in Figure 3-3. Finally, we numerically determine the distribution for the output variable and compare it to the actual output distribution in Figure 3-4.

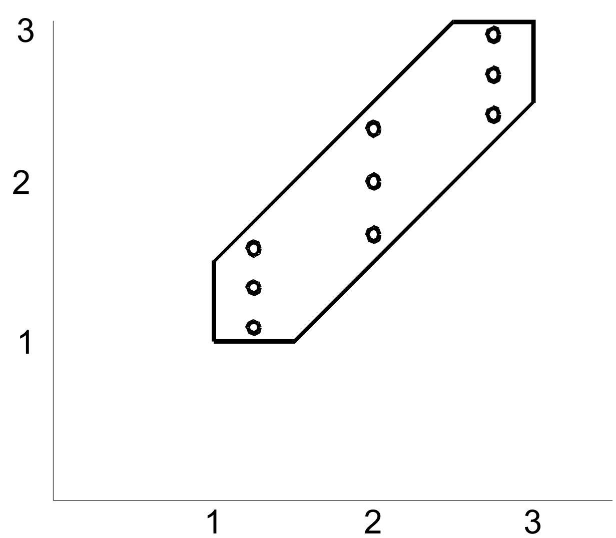
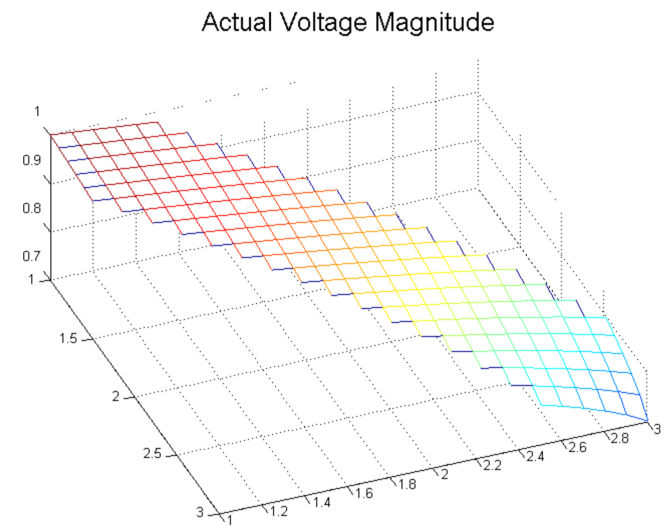


Figure 3-2: Input distribution and PCM points. The parameters (inputs) x and y are distributed uniformly over the polygonal region shown. PCM points are also illustrated.


Actual Voltage Magnitude
1
0.9
0.8
0.7
1
1.5
2
2.5
3
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3

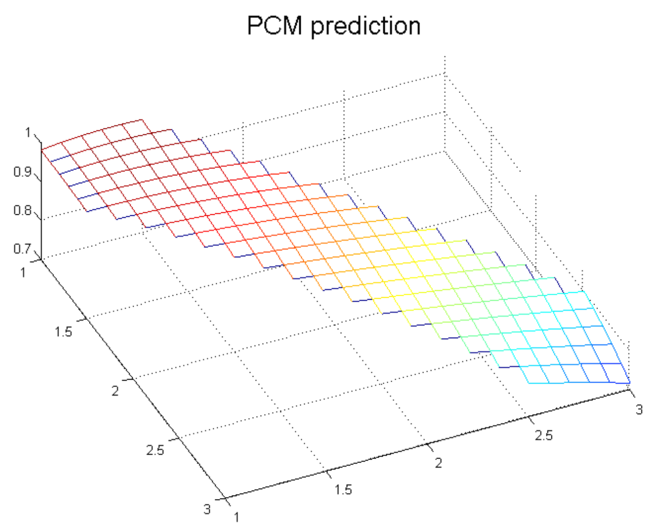
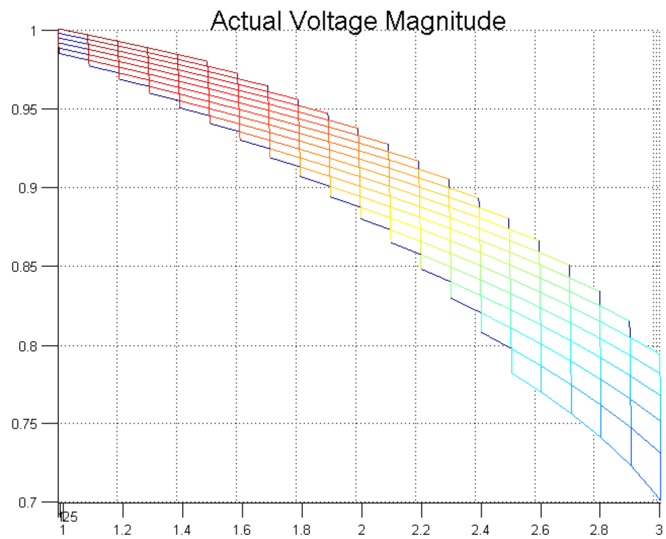

PCM prediction
1
0.9
0.8
0.7
1
1.5
2
2.5
3
1
1.5
2
2.5
3

Figure 3-3(a)



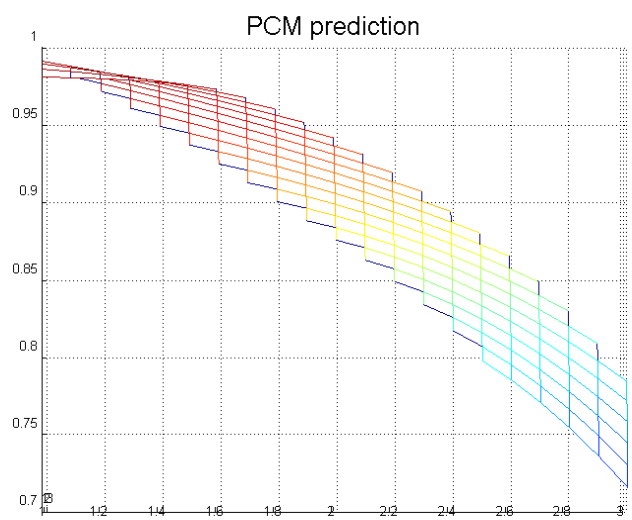


Figure 3-3 (b)

Figure 3-3: Output plots. The mapping between the two input parameters and the voltage output predicted by PCM, compared with the actual mapping. Each three-dimensional mapping is shown from two viewpoints to better illustrate it.

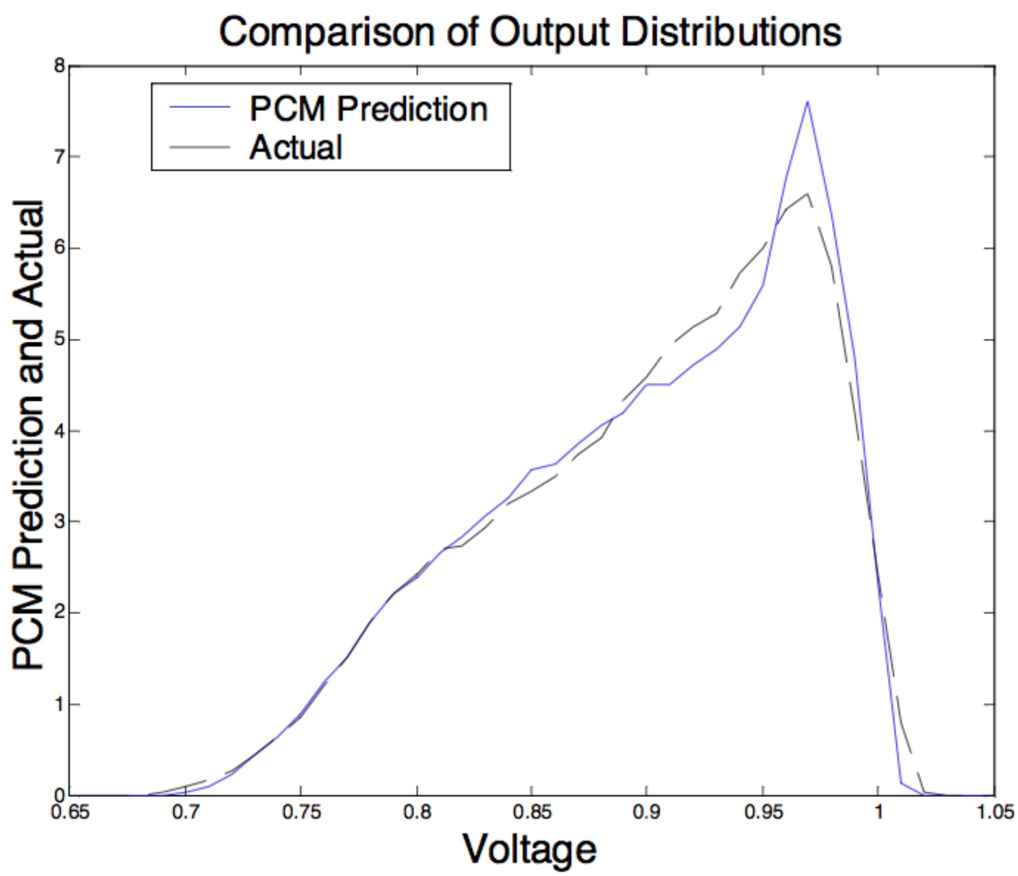


Figure 3-4: The output distribution (i.e., the distribution of the voltage at bus 4) computed from the PCM-based mapping is compared with the actual output distribution.

For this simple example, PCM characterizes both the input-output mapping and the output distribution well. Our solution highlights that the mapping between input and output over the domain of the uncertain loads is non-linear, especially because the correlation between the two loads makes heavy loading conditions frequent. PCM is able to capture this non-linearity, while (in this example) requiring only nine carefully-chosen simulation points to develop a good quadratic mapping. This ability to capture non-linear mappings using only a small number of simulations suggests that PCM holds promise as a PLF algorithm.

We note that the PCM prediction, which requires only nine simulation points, is essentially indistinguishable from the mapping generated through brute-force simulation which we construct using 400 simulations. Thus, our example highlights the significant computational savings that can be obtained through use of PCM.

Finally, we numerically determine the distribution for the output variable and compare it to the actual output distribution in Figure 3-4. We note that PCM is also advantageous in this example, in that we could allow uncertainties with arbitrary joint distributions on the input parameters.

3.4 Relating PCM to the Boundary Load Flow

One difficulty in applying PCM is that the number of required PCM points typically grows exponentially in the number of uncertain parameters. When the number of uncertain parameters becomes large, we note that meshing PCM with a boundary load flow algorithm can provide a tractable solution. In particular, we can select PCM points for a few significant or important uncertain parameters; for each PCM point, we can apply a boundary load flow algorithm with respect to the other uncertain parameters, to find the largest and smallest possible output. Using these extrema outputs, we can develop a pair of mappings from the significant inputs to the output using PCM, which serve as bounds on the actual mapping. Such a meshed algorithm is best illustrated with an example. A plausible alternate description for the load scaling parameters in Figure 3-2 is that these parameters have a strong dependence on a single uncertain input parameter (e.g., temperature) with small, independent deviations from this predicted dependence. For instance, the two parameters could have the form $x = T + \varepsilon_1$ and $y = T + \varepsilon_2$ where T is a significant random parameter, and ε_1 and ε_2 are small, independent random parameters. While we could apply three-dimensional PCM to such a system, a less computationally intensive approach is the following. We can choose PCM points as if we are applying one-dimensional PCM in which the uncertain parameter is T; for each of these PCM points, we can compute the extrema output values over the domain of ε_1 and ε_2 (see [7] for an efficient means for doing so). We can then develop one-dimensional polynomial representations for both sets of extrema. An example of such "boundary mappings" is shown in Figure 3-5.

As far as we know, PCM is the only non-Monte Carlo PLF method that is applicable when inputs are correlated according to arbitrary joint distributions. There is some literature on PLF when the inputs are jointly Gaussian, but we have not seen PLF algorithms for more general correlations among inputs.

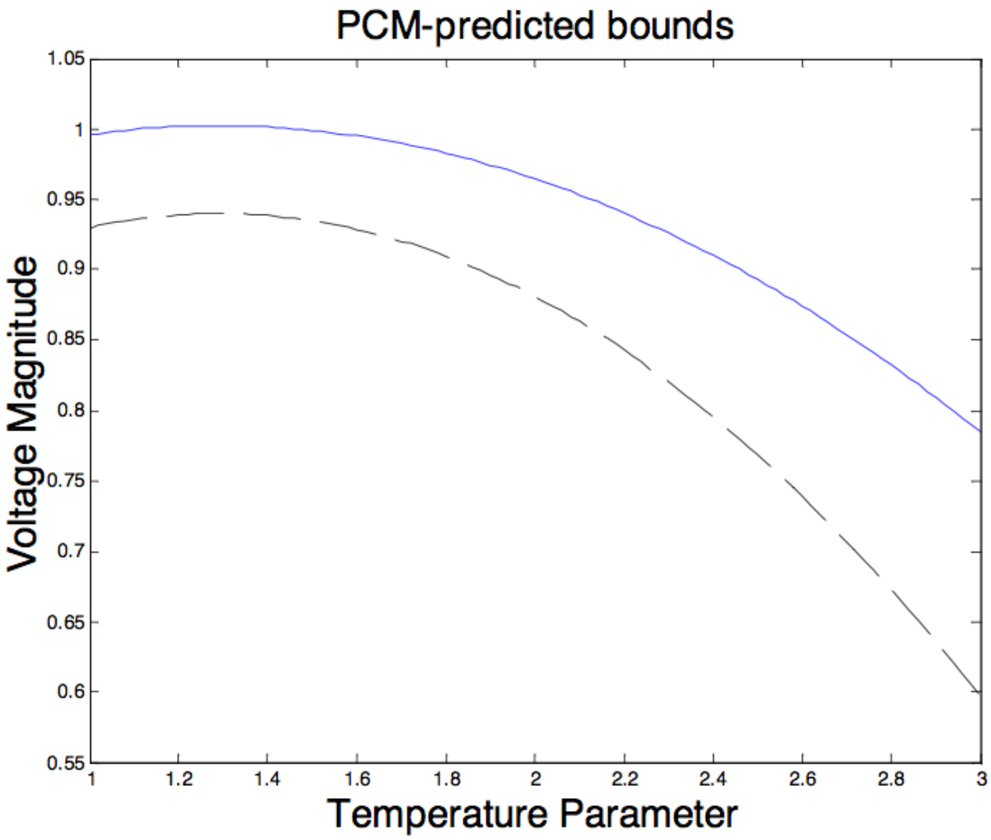


Figure 3-5: Boundary mappings. PCM used with a boundary load flow algorithm to find bounds on the mapping between a significant uncertain parameter and the output voltage. The solid line – upper bound; dashed line – lower bound.

Example: Dynamic time-domain simulation of a disturbance

As reflected in the title of this project, PCM was originally advanced primarily as a tool for evaluating uncertainties in time-step simulations of transient dynamics ([1] and [2]). PCM is indeed valuable for evaluation of uncertainties in transients because it can reduce the number of simulations (which are very often computationally intensive) required for uncertainty analysis; though, as indicated by the examples above, it also has potential for a range of power systems calculations outside of dynamic time-step simulation. When it is applied in the context of dynamic studies, PCM has the advantage that it can be implemented without significant modification of the time-step simulation programs for transients since it only requires measurement of output values for various inputs.

Here we apply two-dimensional PCM to characterize a small power system's transient response to a disturbance. The example that we use is drawn from [6] where it is also used to illustrate the characterization of transient-simulation uncertainties using trajectory-sensitivity methods. Our explorations of this example illustrate how PCM compares with, and complements, the trajectory-sensitivity based methods.

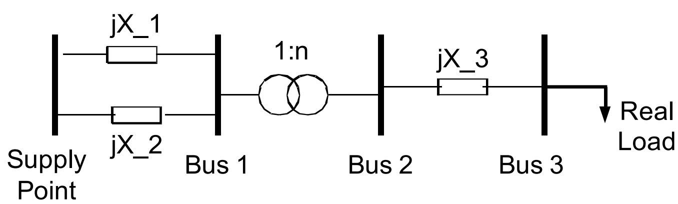


Figure 3-6: Example system for dynamic simulation of disturbance.

We consider the response of this power system to a disturbance, in particular, the tripping of the line with admittance X_1. The uncertain parameters in this example are the load recovery time constant and the tap-changing interval of the transformer.

The small system shown in Figure 3-6 is disturbed through tripping of one of the lines between the supply point and bus *1*. We consider the transient response of the voltage magnitude at bus *3*. This transient response is modulated by the recovery dynamics of the load, as well as the logic of the tap-changing transformer. It is in the parameters of these recovery dynamics that we assume some uncertainty (in accordance with [6]). In particular, we assume that the load time constant T_p and the interval between tap changes T_{tap} are uniformly and independently distributed over the intervals [3, 7] and [15, 25], respectively.

We apply PCM to characterize the mapping between the inputs T_p and T_{tap}, and an output of interest, which we choose to be the minimum voltage on bus *3* during the duration of the simulation. We find that a second-order generalized polynomial model is sufficient to specify the mapping (Figure 3-7). Thus, with only nine simulations, we are able to extract the mapping between the inputs and the output, and further to expose that this mapping is not linear. A compelling feature of PCM is that, using these nine simulations, we can in fact characterize many different output features (e.g., the output voltage at specific times, or various flows in the power network). We note that our analysis compares favorably with the trajectory sensitivity analysis in that we simulate the actual power system rather than a linear approximation thereof. We caution, however, that each simulation of the actual power system may be very expensive computationally as compared to a trajectory sensitivity-based simulation; it is only because so few points are required for PCM that our analysis is feasible. Finally, we mention that one further possible application of PCM to power system dynamic simulations is to identify whether linear relationships between input and output variables hold, and hence to evaluate whether trajectory sensitivity analyses can be used.

3.5 Order Selection Algorithm

In [2], the authors mention the necessity for a good order-selection algorithm for practical applications of PCM. A good order selection algorithm can prove to be cost effective as a new set of simulations is required for each order of PCM polynomial selected.

We observed from our studies on PCM that the order-selection can be done with mere visual inspection in certain cases. Such cases usually involve curves with multiple extrema. But in the case of curves with none or a single extreme, we find the need for a proper order-selection algorithm.

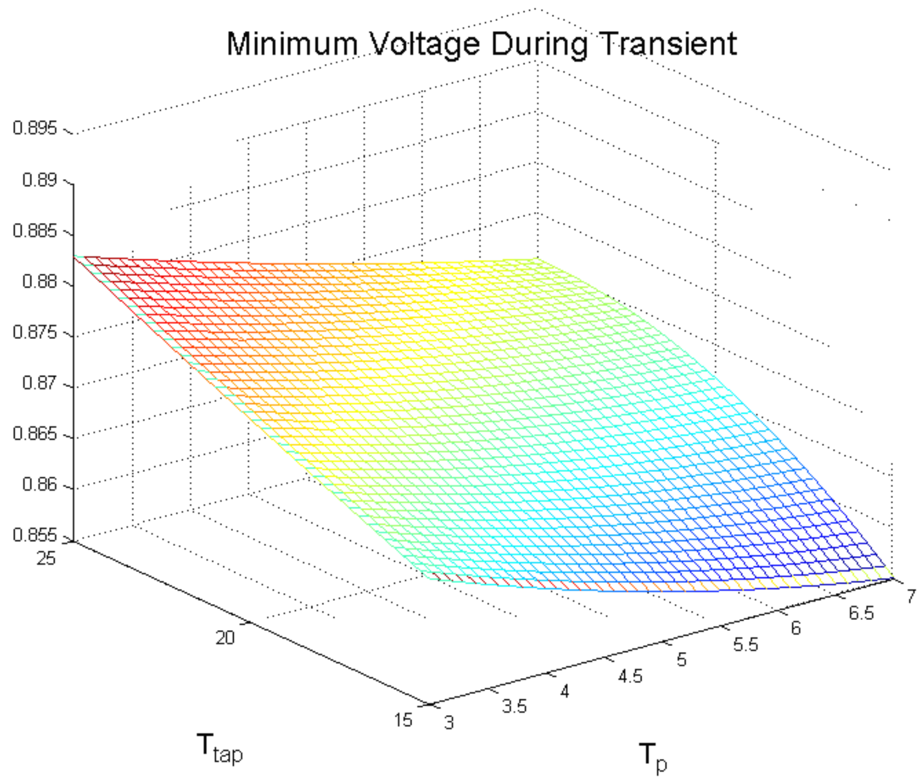


Figure 3-7: PCM generated mapping. The PCM-generated mapping between two uncertain parameters and the minimum voltage reached by bus 3 during a transient simulation.

Before getting into the order selection algorithm we find it worthwhile to define the term **Kullback-Leibler distance.**

The Kullback-Leibler **(**KL) distance gives the distance between two PDFs. In our case the KL distance can be used to compare the distance between successive PDFs (output distributions), and then we can go ahead and select the PDF when the KL distance becomes sufficiently small.

The **Kullback-Leibler** [13] distance is a measure of the difference between two probability density functions P and Q is given by:

$$D(P \| Q) = \int P(x) Log\left(\frac{P(x)}{Q(x)}\right) \tag{3.2}$$

The above integral is finite if and only if P is contained by Q.

THE ALGORITHM:

Our studies suggest the following heuristic order-selection algorithm for one dimensional PCM (applicable to either case mentioned above) followed by the justification of its relevance to higher dimensional PCM.

1. We apply PCM of successive orders (beginning with first-order PCM) until visual inspection suggests that the predicted mapping has not changed between two successive applications.

2. If the mapping predicted by the second-highest-order PCM applied in step 1 has at least two extrema, the visually-determined PCM fit is, in our experience, the proper one. When the mapping has several extrema, we find that the PCM fit converges dramatically to the correct mapping beyond a certain order, so that visual inspection is sufficient to identify the proper fit. Order selection is illustrated for a mapping with three extrema in Figure 3-9.

3. If the second-to-last PCM prediction from the first step has fewer than two extrema, we require an analytical comparison measure to determine whether or not a sufficient order has been chosen. In particular, we numerically compute the output distribution using the mapping of each order. We then compute the **Kullback-Leibler (KL) distance** between successive pairs of distribution (see Table 3-1); if the KL distance between the highest two-order PCM output distributions is sufficiently small (i.e., drastically smaller than the KL distances between lower-order fits), then sufficiently high-order PCM has been used. Otherwise, a higher-order PCM algorithm should be applied until a sufficiently small KL distance is obtained. We note that, if we desire a completely automatic algorithm for order-selection, we can use comparisons of KL distances regardless of the number of extrema.

We applied the order selection to a series RC circuit example from [2], and the results are presented below. The results indicate that the appropriate order is five.

Table 3-1: KL distance comparison

PDF comparison	KL distance
PCM 2nd Vs. PCM 3rd	0.1332
PCM 3rd Vs. PCM 4th	0.1134
PCM 4th Vs. PCM 5th	0.0977
PCM 5th Vs. PCM 6th	0.0033

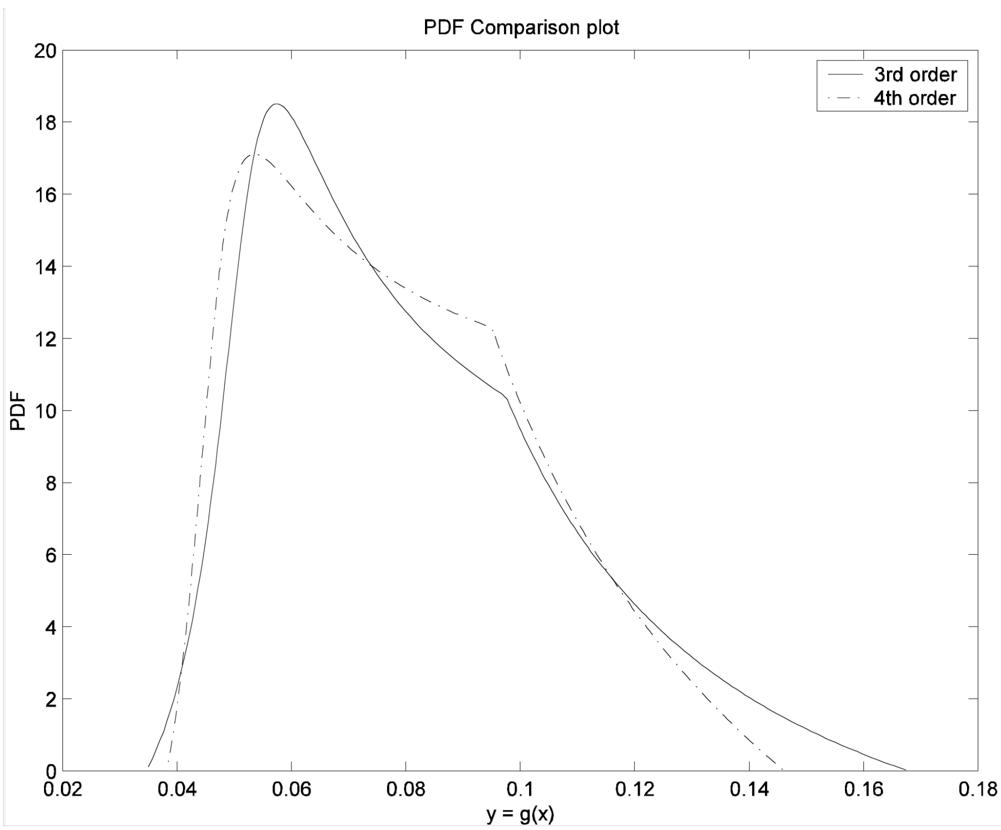


Figure 3-8(a): Comparison of the output distribution plots of PCM generated polynomials of successive order.

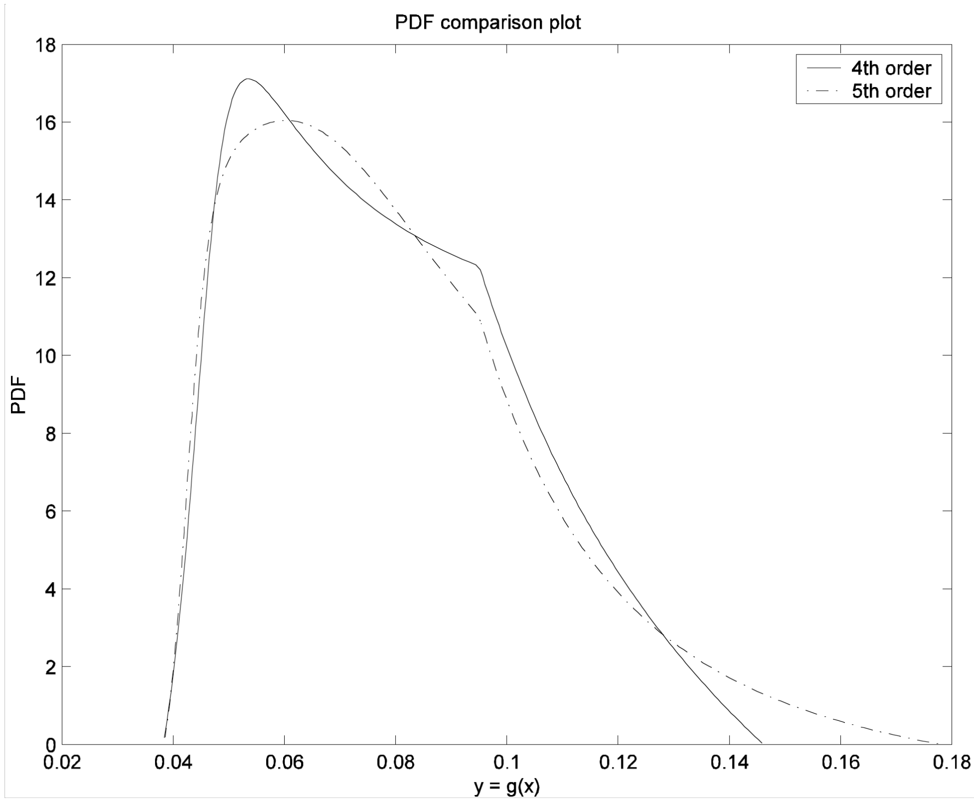


Figure 3-8(b): Comparison of the output distribution plots of PCM generated polynomials of successive order.

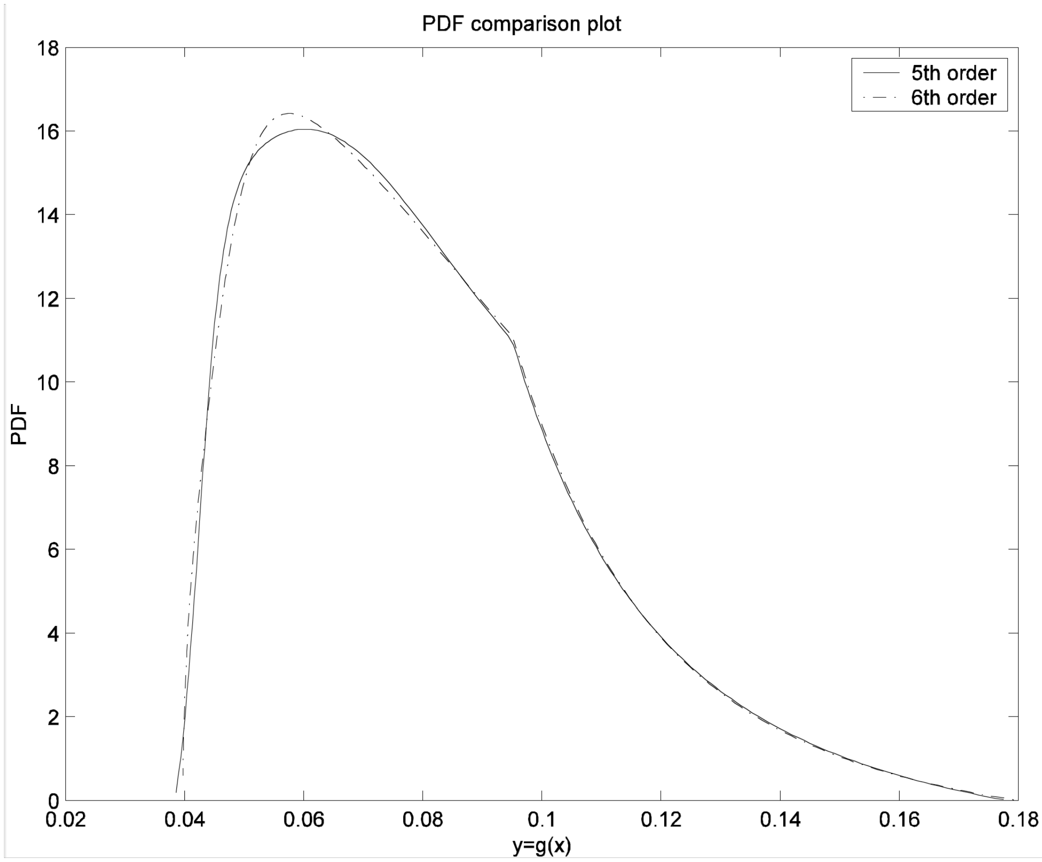


Figure 3-8: Comparison of the output distribution plots of PCM generated polynomials of successive order.

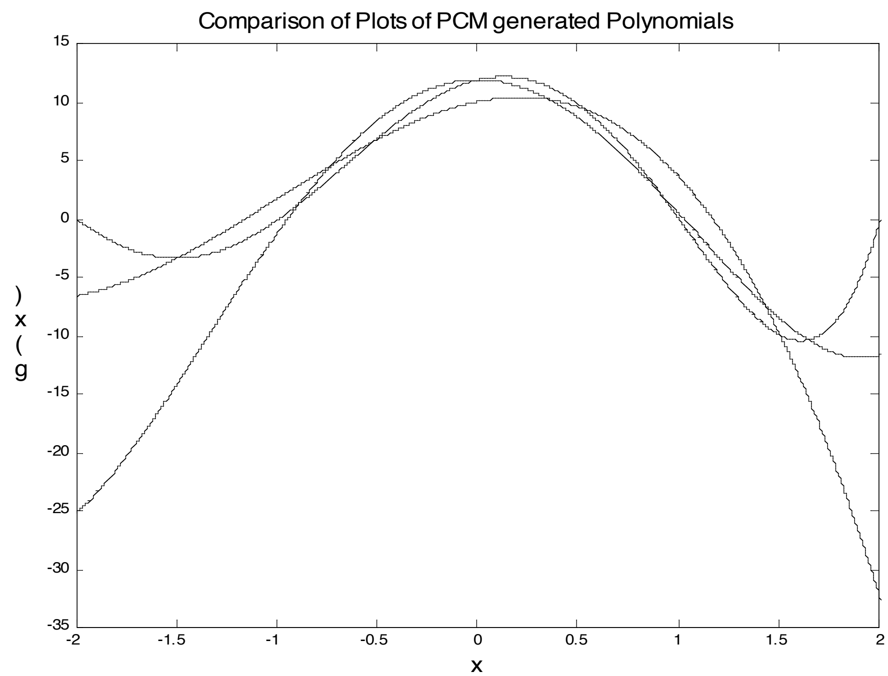


Figure 3-9: Comparison of plots of PCM generated polynomials – two extrema case. Appropriate order can be identified by qualitative features – i.e., the 5th and higher order polynomials correctly have two extrema; lower order polynomials do not.

Our studies show that this algorithm is very effective for single-dimensional PCM. In our two-dimensional PCM algorithm, we generate the PCM collocation points for any one of the uncertain parameters and, on the basis of these points, generate the collocation points for the other uncertain parameter in the system under examination. The order-selection algorithm can be applied when the PCM collocation points for the first parameter are generated. Once the appropriate order is selected, it can be labeled as "the order of the system" and for the second uncertain parameter, we can generate collocation points based on "the order of the system".

4. Optimization

An optimization problem is concerned with finding the minimum or maximum of a function with respect to its arguments which are, in many cases, constrained to a bounded set. There is, of course, a very wide literature on optimization. We refer readers to standard texts, such as [18], for basic notions in optimization.

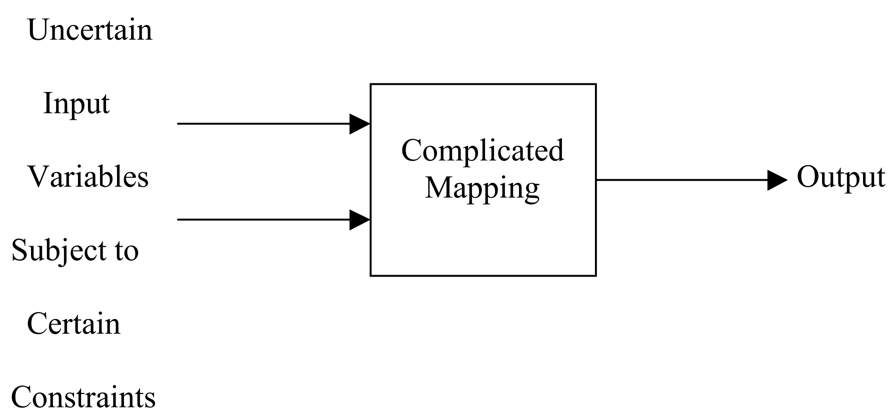


Figure 4-1: Pictorial depiction of the optimization problem

In this section, we consider the problem of optimizing a function over the domain of an uncertain parameter in the case where function evaluations are computationally expensive. In particular, we discuss, in an exploratory manner, the possibility of applying PCM to solving optimization problems.

In general, if the function under study is a black box and if the uncertain input (parameter) variables are continuous and bounded, PCM can be used to approximate the maximum or minimum of the function over the interval. In particular, PCM generates a polynomial approximation for the black box function which can then be optimized.

Broadly, there are two viewpoints on using PCM to find a maximum/minimum.

1. We can address the standard optimization problem of minimizing/maximizing a function over a bounded domain. In this case, it is reasonable for us to assume a uniform distribution for the uncertain parameter in generating the PCM fit.

2. We can view the parameters over which the optimization is done as being uncertain and find the maximum/minimum in a manner that reflects the distribution of theses parameters. That is, by using the distribution in PCM, we can search more carefully for the optimum over high-probability parameter values.

We can come up with a different PCM mapping between the input variables and the output of interest in either case. PCM generates a polynomial mapping from which the minimum or maximum value of the function can be found directly.

An optimization problem in the power systems domain could be maximizing an output voltage at a particular bus over the domain of the input parameters (say, loads).

Illustrative Example

Let us explore this possible application of PCM through an illustrative example. The example used is the five bus load flow example from [10]. Two loads, α_1 and α_2, are uncertain, and the output of interest is the voltage at bus 4. The constraint here is that the sum of the two loads (α_1, α_2) should be equal to 1.75 and α_1 is distributed in the range (0, 1). The optimization problem in this case would be to find the distribution for the loads that maximizes the voltage at bus 4.

Distribution 1:

The PCM mapping was generated by first assuming uniform distribution for α_1 in the range (0, 1) and $\alpha_2 = 1.75 - \alpha_1$.

Distribution 2:

A different distribution was assumed for α_1, viz. $f(\alpha_1) = \dfrac{(3\alpha_1 + 1/2)}{2}$, $\alpha_2 = 1.75 - \alpha_1$, and the corresponding PCM polynomial was generated. The plots of the two polynomials are depicted in Figure 4-2.

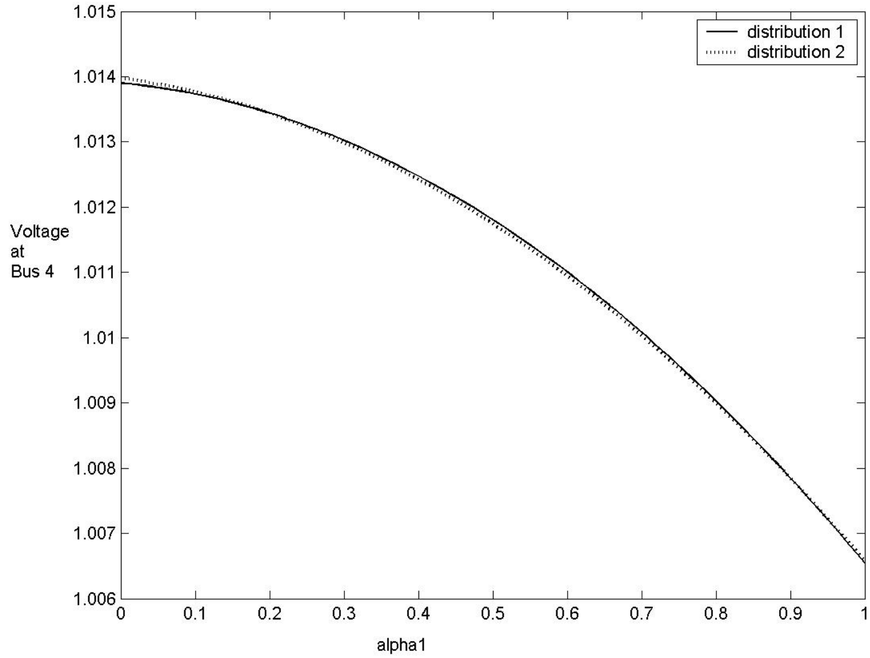


Figure 4-2: Comparison of PCM polynomials generated using the two different input distributions.

The maximum value of voltage for Distribution 1 is 1.0139 at $\alpha_1 = 0$.

The maximum value of voltage for the Distribution 2 is 1.0140 at $\alpha_1 = 0$.

From this we can infer that a higher maximum voltage can be achieved if the loads are distributed as in Distribution 2. The constraint in this problem is that the sum of the loads must equal 1.75. We could also alter the constraints and generate the PCM polynomial for the problem. In either case, we claim that PCM can be a handy tool as it is computationally economic in terms of simulations. Also, once the PCM polynomial is generated, the optimization problem reduces to the task of finding out the maximum/minimum of the PCM polynomial generated.

Accuracy of PCM

In power systems applications one is often interested in determining whether the excursion of key electrical quantities is large enough to trigger breaker action. A key question in applying PCM is simply: how accurate is PCM in capturing the maximum/minimum value of the function?

The following example illustrates the accuracy of PCM in this regard. For Distribution 1 of the previous example, the PCM second order fit is compared with the actual output fit generated via exhaustive simulation.

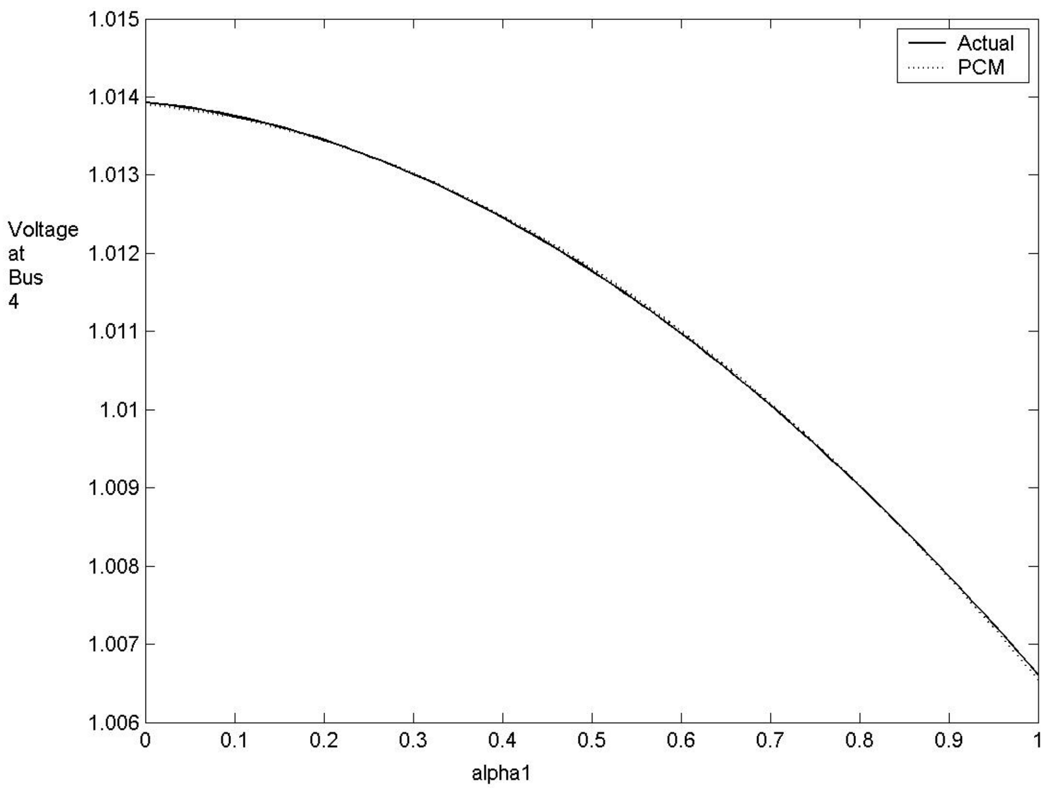


Figure 4-3: Comparison of the actual function generated by exhaustive simulation with the PCM generated polynomial using Distribution 1.

To the accuracy discernible in the figure, the two plots cannot be differentiated; this shows that PCM is quite accurate in modeling the function and eventually the minimum/maximum of the function. The maximum value is captured accurately up to 4 decimal places. It is the same for both the fits at a value of 1.0139.

4.1 Advantages in using PCM

The main advantage again is computational economy. PCM requires a small number of simulations for generating the polynomial mapping. PCM also specifies a low-order mapping that approximates a much higher-order relationship. For instance, if the original relationship is a quartic, a quadratic PCM polynomial provides a good approximation in many cases. The example depicted by the figure below illustrates this fact.

<u>Example:</u>

The parameters are x_1 and x_2. The distribution and constraints are as follows:

$$-1 < x_1 < 2, \text{ Uniform distribution.}$$

$$x_2 = 2 - x_1 .$$

The output function is $g(x_1, x_2) = x_1^4 - 2x_1x_2^2 + x_2 + x_1 - 4$.

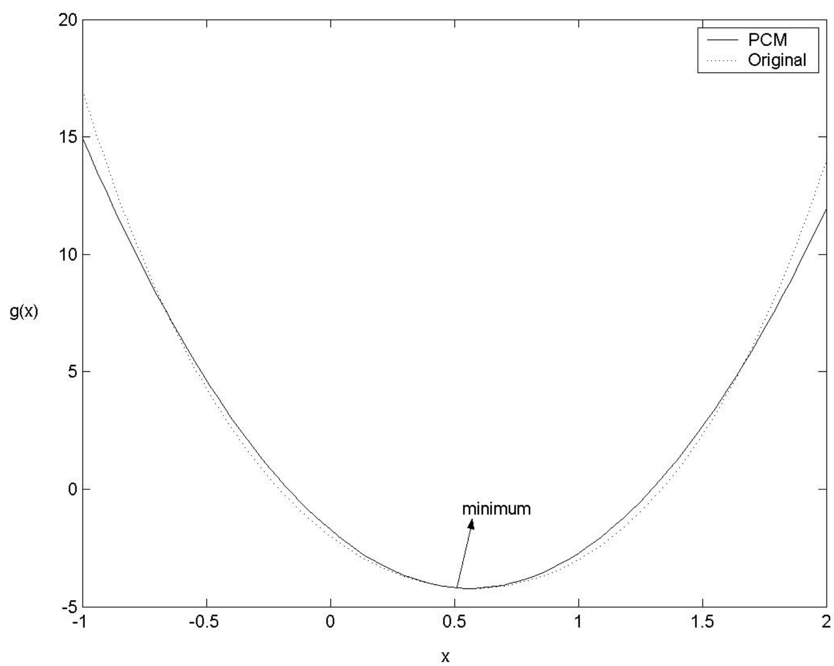


Figure 4-4: The minimum value of a quartic function captured by a PCM quadratic polynomial.

The PCM mapping shown in Figure 4-4 for this relationship is a quadratic whereas the actual relationship is a quartic, yet it captures the minimum accurately. In case there is more than one minimum/maximum, the lower order PCM mapping could possibly capture one of them.

The capability of PCM to solve optimization problems adds a new dimension to the algorithm. We have shown that PCM could be effectively used for maximizing voltage in power systems. This is just a rudimentary attempt at using PCM to solve optimization problems. To tout PCM as an optimization algorithm in general is not appropriate. However, the results obtained so far are promising and we hope that future work in PCM will be concentrated in this area.

4.2 Comparing PCM with a Traditional Minimization Technique

It is interesting to see how PCM fares when pitted against traditional function minimization techniques, such as the steepest descent, gradient descent, or the Newton-Raphson method.

Among the above mentioned techniques, Newton-Raphson typically offers the fastest convergence rate. Newton-Raphson is an iterative process for minimizing a function with respect to one or more variables. The Newton-Raphson formula for minimizing a single variable (one dimensional) function $f(x)$ is

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

The iteration is started by guessing an initial value x_0.

The method iteratively tries to locate the minimum of the function. The accuracy of the technique increases with the number of iterations. Of course, the initial guess must be intelligent otherwise this technique may not converge.

When comparing an iterative minimization technique like Newton-Raphson with PCM, we must first identify a yardstick for comparison. Comparing the number of iterations that Newton-Raphson takes to converge to the minimum value of the function with the order of the PCM that produces the polynomial with the correct minimum value appears sensible. For each Newton-Raphson iteration, we need to calculate the value of the first and second derivative of the function under study at the current estimate for x_{n+1}. Thus, we need to simulate the function iteratively; as a matter of fact, we would be performing more than one simulation per iteration as we need to calculate both the first and second derivative of the function each time.

Example

Minimize $g(x) = x^2(x^2 - 1), -1 < x < 2$

The above function can be minimized analytically. The purpose of choosing such an example is that it makes the task of comparison easier. We can compare the actual minima calculated analytically with those computed using Newton-Raphson and PCM.

The function has a local minimum at $x = 0$ and global minima at $x = \left\{ \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$. The minimum value of the function is -0.25.

We applied Newton-Raphson and PCM to minimize this function. Uniform distribution was assumed for the variable x. The results are presented in Table 4-1.

Table 4-1: Comparison of PCM with Newton-Raphson minimization

Newton-Raphson Iterations	PCM order
1st Iteration $x_{\min} = 0.5, g(x_{\min}) = -0.1875$ 2nd Iteration $x_{\min} = 1, g(x_{\min}) = 0$ 3rd Iteration $x_{\min} = 0.8, g(x_{\min}) = -0.2304$ 4th Iteration $x_{\min} = 0.721127, g(x_{\min}) = -0.249599$ 5h Iteration $x_{\min} = 0707505, g(x_{\min}) = --0.24999968$ 6h Iteration $x_{\min} = 0.707107, g(x_{\min}) = -0.25$	1st Order $x_{\min} = -1, g^*{}_{\min} = --0.750011$ 2nd Order $x_{\min} = \text{-}1.07427, g^*{}_{\min} = \text{-}0.578827$ 3rd Order $x_{\min} = \left\{0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, g^*_{\min} = \{0, -0.25, -0.25\}$ 4th Order $x_{\min} = \left\{0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, g^*_{\min} = \{0, -0.25, -0.25\}$

Observations

Newton-Raphson took six iterations to converge to the minimum. This is partly due to the initial guess being slightly off the hook. If we had taken $x_0 = 1$, we would have gotten a convergence in five iterations.

Although the function under study is a quartic, the PCM cubic polynomial captures the minimum accurately. We could have stopped with five iterations for Newton-Raphson and at the 3rd order for PCM. The extra iteration (and order) were just to check for convergence.

Assuming we had taken $x_0 = 1$ and did not consider the extra iteration to check for convergence, then Newton-Raphson estimates the minimum in four iterations. In the same vein, not considering the extra PCM order, the 3rd order PCM polynomial captures the minimum of the function and it took four simulations for generating the 3^{rd} order polynomial.

Hence, in this example, PCM performs as well as Newton-Raphson, but this may not always be the case. Our purpose was to illustrate that PCM could be used as a tool for minimization and not to claim or try to prove that it works better than existing minimization techniques. PCM has a long way to go in this aspect and this chapter is just prefatory to the study of using PCM to solve optimization problems.

5. Information-Theoretic Approach to Parameter Reduction in PCM

The advantage of PCM over traditional Monte Carlo simulation techniques is that PCM requires very few simulations to identify the mapping between the uncertain input(s) and the output of interest. Although PCM is computationally economic, PCM too requires an exponential amount of simulations as the number of inputs increases. For instance, if k is the number of system uncertainties, it would take $(n+1)^k$ simulations to generate a polynomial of order n. Though this number of simulations is typically small compared to the number needed for traditional Monte Carlo techniques, it is necessary to come up with variable reduction techniques to make the process of modeling the system uncertainties less cumbersome.

When the multiple PCM inputs are strongly correlated, the input variables potentially carry a lot of redundant information. In such cases it may be possible to model the mapping using only a subset of the input variables or a lower order basis for them. In order to do so, some mechanism for measuring dependencies between the input variables is required. Some interesting information-theoretic concepts, including Entropy and Mutual Information, can be used to measure dependencies between the variables. In particular, we use Mutual Information as a good measure of dependency between jointly distributed random variables for which variable reduction in PCM can be achieved. The remainder of this chapter discusses this information-theoretic approach for reducing the number of input variables in PCM.

5.1 Dependency Measurement

Before describing its application to PCM, we review certain information-theoretic concepts useful for the study.

Mutual Information [13] is an information-theoretic concept which can be used as an indicator for the degree of dependency between jointly distributed random variables. Another useful measure is the correlation coefficient which gives the degree of correlation between two random variables; correlation is the degree to which two or more quantities are linearly associated [24].

The **Differential Entropy H(X)** [13] of a continuous random variable X with a density f (x) is defined as:

$$H(X) = -\int_S f(x)\log_2 f(x)dx \tag{5.1}$$

where S is the support set of the random variable. The set of x for which f(x) > 0 is called the support set of x.

Entropy [13] is a measure of randomness of a random variable, and, in the discrete domain, it represents the shortest description length (in bits) of the variable.

Differential Entropy [13] is also related to the shortest description length. One caveat here is that we can get negative values for differential Entropy. Hence, an appropriate measure for description length is the volume of the support set of the random variable given by $2^{h(X)}$, which is obviously non-negative.

The **Joint Entropy H(X; Y)** [13] of jointly distributed continuous random variables x and y with joint density f (x, y) is defined as:

$$H(X;Y) = -\int f(x,y)\log_2 f(x,y)dxdy \tag{5.2}$$

The Joint Entropy again is a measure of randomness or description length. The difference here is that we are considering a vector of random variables instead of a single random variable.

Mutual Information

The **Mutual Information I (X;Y)** [13] between two jointly distributed continuous random variables x and y with joint density f (x,y) is defined as:

$$I(X;Y) = \int f(x,y)\log_2\left(\frac{f(x,y)}{f(x)f(y)}\right)dxdy \tag{5.3}$$

The Mutual Information between two jointly distributed random variables is the amount of information one random variable contains about another. In a sense, it is the reduction in uncertainty of the random variable X due to the knowledge of Y, and vice versa. It is an estimate of the strength of association between jointly distributed random variables.

Correlation Coefficient

The **Correlation Coefficient** [13] is a numeric measure of the strength of a linear relationship between two random variables. It is given by the equation

$$\rho(x, y) = \frac{\operatorname{cov}(x, y)}{\sqrt{\sigma^2(x)\sigma^2(y)}} \tag{5.4}$$

where cov (x, y) is the covariance defined as:

$\operatorname{cov}(x, y) = E\left[(x - E(x))(y - E(y))\right] = 0,$ if x and y are independent

E = Expected value

σ^2 = Variance.

The correlation coefficient lies between -1 and 1. It is -1 if x and y are perfectly negatively correlated. It is +1 if x and y are perfectly positively correlated.

5.2 Reducing the Number of Input Variables

Reducing or filtering input random variables is the process of eliminating certain variables considered containing redundant information, and using the remnant variables for generating the PCM mapping between the inputs and the output of interest.

For instance, if x and y are the input random variables, the 2-D PCM fit for the system will be of the form:

$$g(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x^i y^j \tag{5.5}$$

Assuming that y has redundant information and can be eliminated, the process of developing the PCM mapping for the system degenerates to an 1-D PCM problem with only one input random variable, viz., x. The PCM fit after variable reduction will be of the form:

$$g^*(x) = \sum_{i=0}^{n} a_{i\,j} x^i \tag{5.6}$$

Though we discard the random variable y, it must be noted that we need both random variables for running simulations of the black box system under analysis. We suggest the usage of the conditional mean for the redundant random variable instead of its PCM values.

The caveat to be kept in mind is that the reduction should not result in information loss; that is, $g^*(x)$ should approximate $g(x,y)$ well. The moments of both the polynomials and their output distribution plots are good comparative measures that can be used to check the accuracy of the reduction process.

The combination of Mutual Information and Correlation Coefficient values can be used as a tool for deciding when to reduce input random variables.

We distinguish between two cases in which we can eliminate input random variables. We will call them Case I and Case II. In the following section the two cases are described via examples.

Case I General Characteristics

In Case I, **variables** have high Mutual Information between them. Experimental results suggest that a reasonable cutoff value for the Mutual Information measure is 3.5 and greater. The variables are also strongly positively or negatively correlated; i.e., their correlation coefficient value is close to +1 or –1. Experimental evidence suggests that a good cutoff value is +0.9 and greater (or – 0.9 and lower).

Case II General Characteristics

In Case II, variables do not have a very high Mutual Information value, but the Entropy of one of the input variables is very small compared to the Entropy of the other random variable and also to the Mutual Information value. The Correlation Coefficient is not significant in this case but the variance of the individual random variables can be used for comparison.

In either of the two cases, it is not necessary for the reduced fit to be of the same order as the original. We recommend using a higher order for the reduced fit for the sake of accuracy.

Numerical Results for Examples

Case I Numerical Details

One of the random variables is uniformly distributed and the distribution for the second random variable exhibits a very strong dependence on the first. The motivation for selecting an example where the variables exhibit strong correlation is to show that the Correlation Coefficient is related to the notion of Mutual Information.

$$1 < x < 2$$

$$x - 0.03 < y < x + 0.03$$

$$f(x, y) = (1/0.06)$$

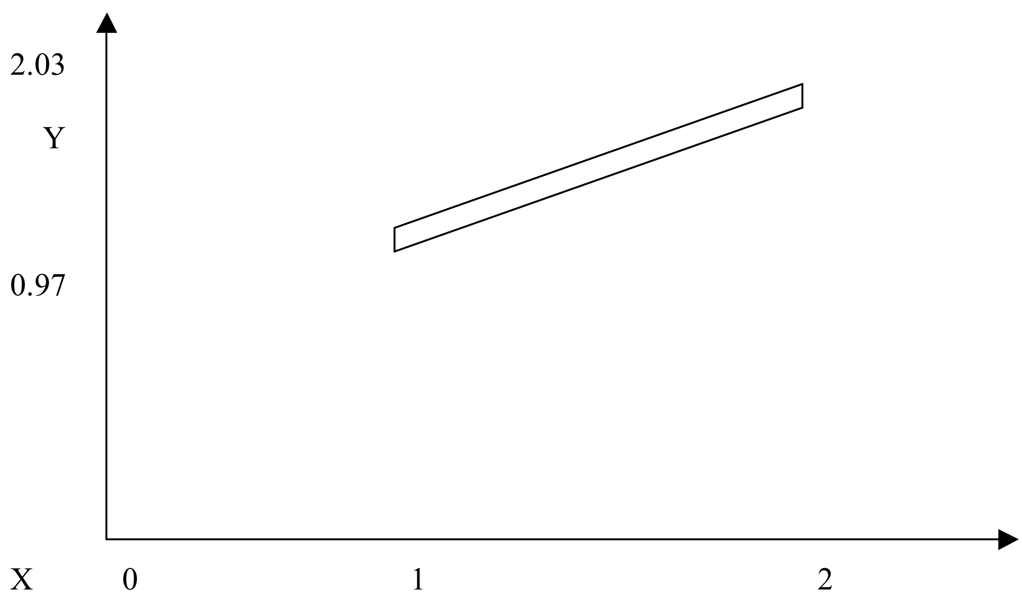


Figure 5-1: Input distribution region.

The Mutual Information, Joint Entropy, and Correlation Coefficient values are:

$I(X;Y) = 4.24261$

$H(X;Y) = -0.246373$

$\rho(x, y) = 0.998205$

We assume that the two random variables are two load-scaling parameters chosen for the 5-Bus load flow example from [10]. For details, please refer to earlier chapters in the report. As before, loads 4 and 5 are considered uncertain and the output of interest is the voltage at Bus *4*.

The 2-D PCM quadratic polynomial mapping was generated for this example. The 1-D PCM Quadratic polynomial was then developed by using PCM points for only one of the input random variables (viz., x) while for the second variable, y, the conditional mean at each x value was used. The 2-D and 1-D polynomials and their distributions are respectively:

<u>2-D</u>

$$g(x,y) = 2.86\times10^{-13}x^2y^2 - 9.22\times10^{-13}xy^2 - 0.0014x^2y + 0.015xy + 6.07\times10^{-13}y^2$$
$$-0.029y + 0.026x^2 - 0.082x + 1.09$$

$$E(g(x,y)) = 0.960256$$

$$Variance = 0.000634529$$

<u>1-D</u>

$$g(x) = -0.0233x^2 - 0.017x + 1.04$$

$$E(g(x)) = 0.960256$$

$$Variance = 0.000633957$$

Thus, we see that the Expected Values and Variances agree strongly. Next, the output distribution of the 1-D and the 2-D PCM polynomials are compared in Figure 5-2.

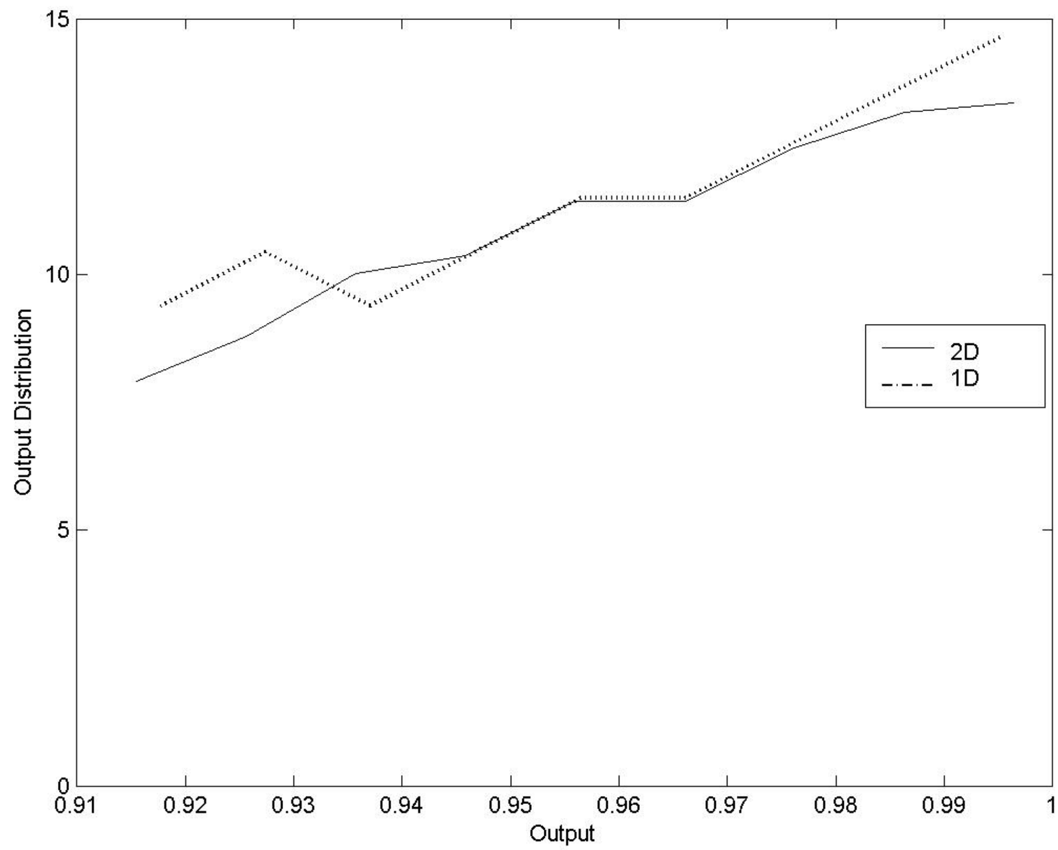


Figure 5-2: Comparison of output distributions, Case I. The solid line represents the output distribution based on the 2-D PCM approximation and the dotted line represents the output distribution based on the 1-D PCM approximation.

Figure 5-2 depicts the plots for the output distribution corresponding to the 2-D and 1-D PCM. The output distribution plots are quite similar, corroborating the statistical results presented before. The results show that when the two random variables have a high Mutual Information value and are strongly correlated, it is sufficient to use just one of the random variables for characterizing the input-output mapping. Table 5-1 shows how the Mutual Information increases as the size of the distribution is reduced in the y-direction.

Table 5-1: Mutual Information values

Distribution	I(X; Y)
$x - 0.1 < y < x + 0.1$	2.31276
$x - 0.09 < y < x + 0.09$	2.40428
$x - 0.08 < y < x + 0.08$	2.52115
$x - 0.07 < y < x + 0.07$	2.67316
$x - 0.06 < y < x + 0.06$	2.87623
$x - 0.05 < y < x + 0.05$	3.15829
$x - 0.04 < y < x + 0.04$	3.57336
$x - 0.03 < y < x + 0.03$	4.24261
$x - 0.02 < y < x + 0.02$	5.51107
$x - 0.01 < y < x + 0.01$	8.97349

The results indicate that the correlation gets stronger, as one would expect, because the dependence of y on x becomes stronger. Mutual Information captures this phenomenon effectively.

Case II Numerical Details

$$f(x, y) = \left(\frac{x}{3}\right) - \left(\frac{y}{9}\right), 1 < x < 3, 1 < y < 2$$

$$I(X;Y) = 0.000431909$$

$$H(X) = 0.615262$$

$$H(Y) = -0.00206016$$

$$Variance(x) = 0.283951$$

$Variance(y) = 0.0829904$

Again the same 5-Bus load flow example was used with the voltage at bus *4* as the output of interest. Both 2-D and 1-D PCM fits were generated in the same fashion as in Case I.

The results are:

<u>2-D</u>

$$g(x,y) = -0.0017x^2y^2 + 0.004xy^2 + 0.0022x^2y - 0.011xy - 0.011y^2 - 0.0001y$$
$$- 0.007x^2 - 0.016x + 1.04$$

$Mean = 0.922204$

$Variance = 0.0011$

<u>1-D</u>

$$g(x) = -0.00776x^2 - 0.0024x + 1.017$$

$Mean = 0.923196$

$Variance = 0.00092$

Distribution Plots:

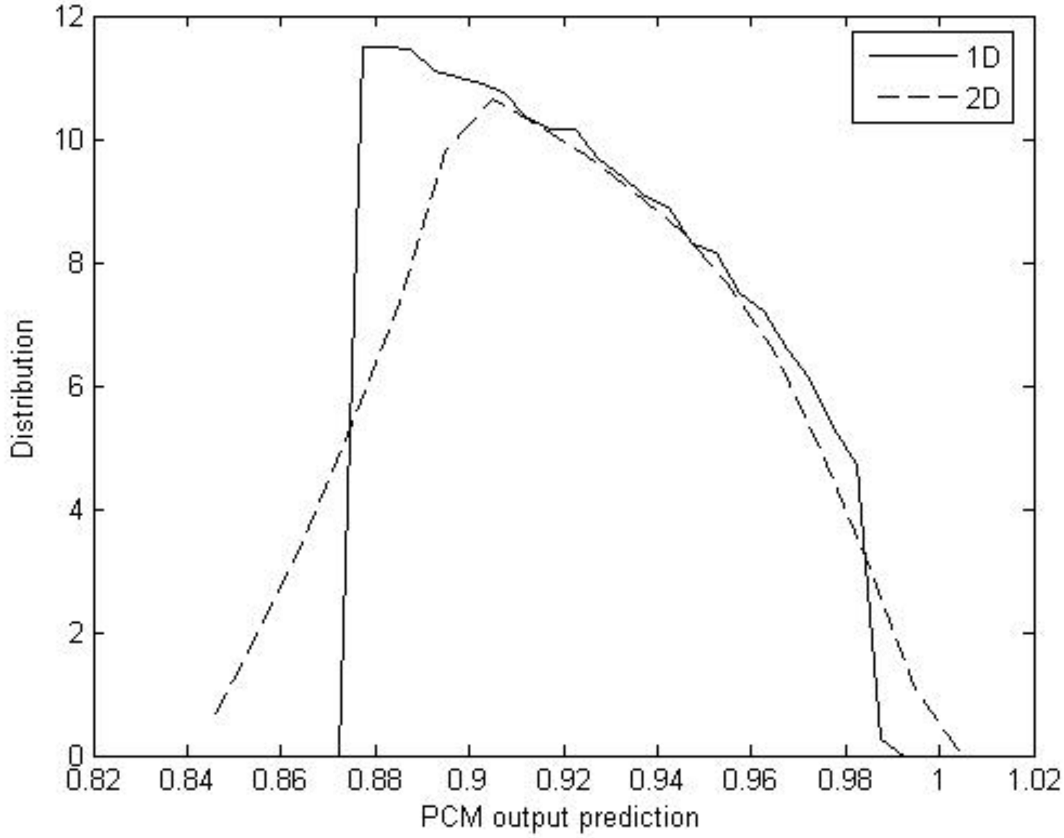


Figure 5-3: Comparison of output distributions, Case II. The solid line represents the output distribution based on the 2-D PCM approximation and the dotted line represents the output distribution based on the 1-D PCM approximation.

The expected values and variances agree well, and the distribution plots are similar. Thus, the results support the claim that we can use just one of the random variables instead of two to model the input-to-output mapping of the system when the Mutual Information and Entropy values of the input variables are as described under the conditions for Case II.

5.3 Justification

In our development, we have suggested using Mutual Information as the primary criterion for eliminating redundant random variables, and have mentioned that the Correlation Coefficient can provide a second criterion. A brief comparison of the two criteria is valuable for identifying the advantages and limitations of each. In this section, we provide a conceptual comparison of the two.

Broadly, our motivation for invoking information-theoretic concepts rather than only using the Correlation Coefficient is that less restrictive criteria for parameter reduction can be developed. Specifically, by eliminating parameters with high mutual information, we permit elimination of parameters that are nearly deterministically but non-linearly related. For instance, consider the following, which is a limiting case, in that one parameter is a deterministic function of the other:

<u>Example Comparing Mutual Information versus Correlation Criteria:</u>

Consider a system with a pair of uncertain inputs X_1 and X_2, $X_2 = X_1^2$, and X_1 is uniformly distributed between -1 and +1. Since X_2 is a deterministic function of X_1, the mapping between X_1 and the system output is a deterministic one. Hence, we can identify the mapping between X_1 and the output using PCM, albeit perhaps with a higher-degree polynomial than if the output is expressed in terms of both X_1 and X_2. Hence, our criterion should eliminate X_2 (or alternatively X_1) in this case. Since the conditional entropies of each variable given the other are arbitrarily negative for this pair of random variables, the information-theoretic condition indeed indicates that one of the parameters can be eliminated. However the Correlation Coefficient for X_1 and X_2 is zero, and, hence, a correlation-based test would not indicate that a parameter could be eliminated.

To summarize, information-theoretic concepts allow us to eliminate parameters that are strongly-interdependent in non-linear ways, while Correlation Coefficients only allow us to identify linear dependencies. Since the applicability of PCM is based on whether or not the mapping from the parameters to the output is deterministic rather than on its linearity, the less restrictive information-theoretic condition should be the primary one. It is worth noting that a high correlation coefficient yields a stronger result in that it indicates not just the possibility for parameter reduction but the possibility for using a lower-order PCM fit of the same degree.

Criterion Considered: I(X;Y), var(X) or **I(X;Y)-H(X,Y)**

In the above development, we have distinguished between two cases – one in which high mutual information permits elimination of variables, and another in which the low spread (variance) of one of the variables permits its elimination. The reader may wonder why these measures cannot be combined into a single one (e.g., why Mutual Information by itself cannot be used to eliminate variables), and hence some further discussion of the criteria is needed. In fact, the underlying difference between these two cases brings up a more general concern about what the proper criterion is and suggests yet another measure for variable reduction.

Perhaps the best way to explain the distinction between the two cases is to note that the Mutual Information and Correlation Coefficient are unitless quantities, while Entropies and variances have units. That is, simple scaling of the random input variables does not change their mutual information, but does change the variance and Entropy of each variable. Thus, the Mutual Information (or Correlation Coefficient) identifies the reduction in uncertainty in one variable through knowledge of the other, but does not identify the actual randomness in these variables. Thus, when we use the mutual information-based criterion, we are considering the reduction in one variable's uncertainty due to knowledge of the other in a *scale-free way*. In contrast, when we choose to eliminate random variables with small variances, we make the assumption that the two variables are defined on the same scale, and that variation in the output of interest over equally-sized domains of each variable are on the same order. Such an assumption is reasonable, for instance, in the power flow example in which the random parameters are scaling factors for loads of nearly the same magnitude and hence also have comparable impact on the output voltage.

More generally, when the absolute scaling of the system is well understood, we note that the mutual information-based criterion can be modified to take into account the absolute statistics of the inputs. One way of doing so is to use a measure such as

$I(X;Y)-H(X,Y)=-H(X|Y)-H(Y|X)$. When this quantity is sufficiently positive, the relative Entropy of X given Y and/or the relative Entropy of Y given X are small, and reduction of one of the variables is possible. We note that this measure accounts for the absolute uncertainty in each variable conditioned on the other rather than using only the change in uncertainty.

6. Method Illustration in IEEE 14-Bus Power System Example

In this chapter, we will give the results of applying several of the concepts discussed thus far to a larger electric power systems example. Although stylized examples have been provided in each chapter, a larger realistic example is necessary to demonstrate the applicability of the technique.

Power systems loads are generally classified as industrial, commercial or residential based on the usage sector. Traditionally loads classified under the same category have interdependencies and if these interdependencies are strong enough, we can use the techniques discussed in this report for generating a reduced-order PCM polynomial for the system.

Figure 6-1 represents an IEEE 14 Bus Test System. The numbers inside the squares represent the transmission line numbers. and the bus numbers are encircled.

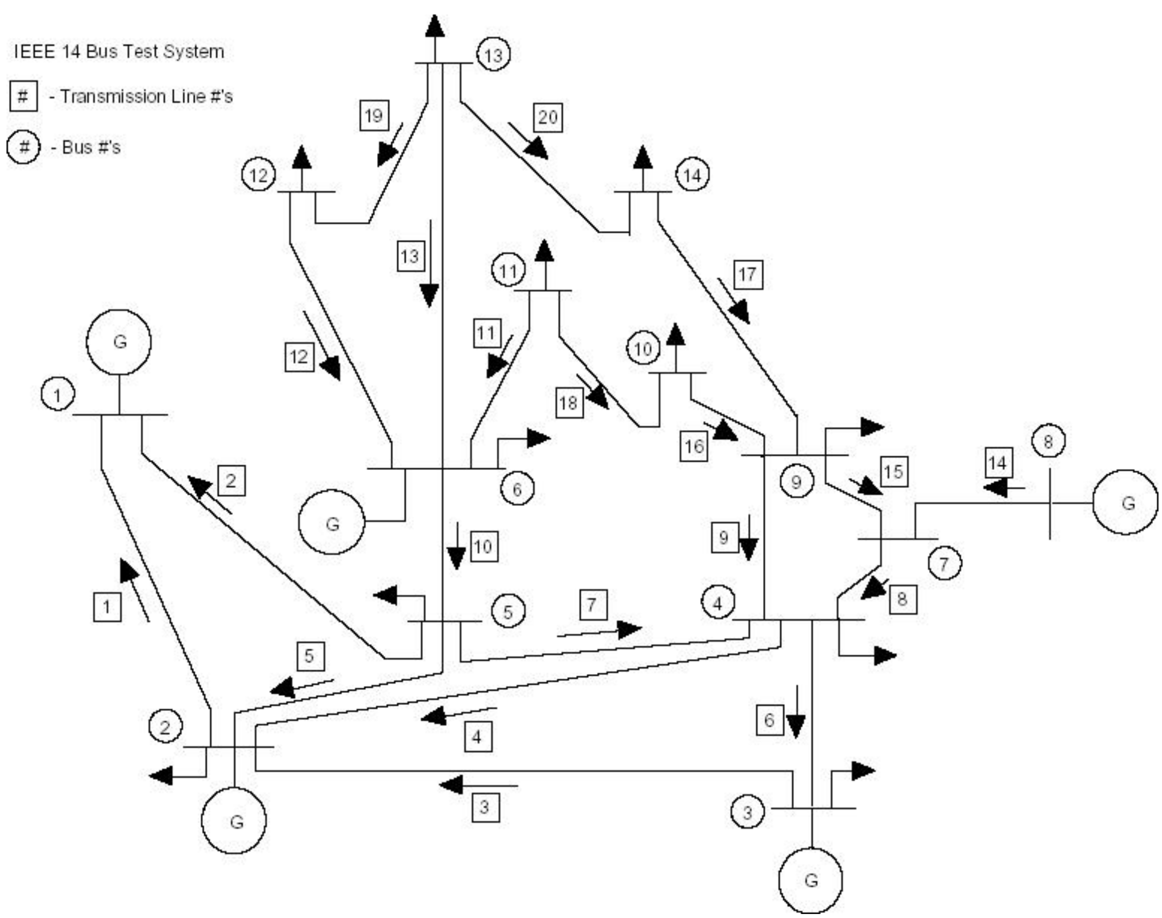


Figure 6-1: IEEE 14 bus test system

For our purpose, we have assumed that the loads at six of the buses are uncertain, and we divide the uncertain loads into two categories, viz. industrial and commercial. To be specific, Loads at buses *4*, *5* and *9* are categorized as industrial whereas the loads at buses *12*, *13* and *14* are considered commercial. We have two sets of three uncertain parameters and our output of interest is the magnitude of the voltage at bus *4*.

We will approach the example as follows. First, we will generate a PCM linear fit for the mapping between the uncertain loads and the voltage at bus *4*. Since there are six uncertain parameters, to come up with a PCM linear polynomial we would require 64 system simulations. Then, by applying the information-theoretic techniques discussed earlier, we attempt reduce the number of uncertain loads from six uncertain loads to just two uncertain loads. The rationale here is that under each load type we have assumed two of the loads to be strongly dependent on one predominant load; i.e., load at bus *4* in the case of industrial and the load at bus *12* for commercial. As a result, they would have sufficiently high Mutual Information and Correlation Coefficient values for us to reduce the number of input uncertain parameters. After uncertain parameters reduction, we attempt to model the input-output mapping using only the above mentioned two predominant loads. The details of the load distributions are given below in Table 6-1. For ease of mathematical representation, we label the loads at buses *4, 5, 9, 12, 13, 14* as a, b, c and x, y, z respectively.

Table 6-1: Load details, and Mutual Information and Correlation Coefficient values.

Distribution details
Industrial Loads $0.1 < a < 0.6$ Uniformly distributed $a - 0.03 < b < a + 0.03$ Uniformly distributed $a - 0.02 < c < a + 0.02$ Uniformly distributed $I(A;B;C) \approx 17.7564, H(A,B,C) \approx -0.103064, Cov(a,b) \approx 0.998847, \mathrm{Cov(a,c)} \approx 0.999418$ Commercial Loads $0.25 < x < 0.7$ Uniformly distributed $x - 0.04 < y < x + 0.04$ Uniformly distributed $x - 0.03 < z < x + 0.03$ Uniformly distributed $I(X;Y;Z) \approx 9.44264, H(X,Y,Z) \approx -0.112934, Cov(x,y) \approx 0.99, Cov(x,z) \approx 0.99$

Results

Table 6-2: Load Flow Mean and Variance Comparisons in 14 Bus Example

With 6 uncertain loads	**With 2 uncertain loads**
Mean = 0.9984 Variance = 0.00016829	Mean = 0.997118 Variance = 0.000166101

The mean and variance values agree very well with one another. This numerical result confirms that the reduction is successful.

7. Summary and Future Work

In this report we have explored the Probabilistic Collocation Method (PCM) and its potential applications in the power systems context, seeking to demonstrate that it is an computationally effective technique for simulating and modeling outputs of complex mappings that underlie many power systems phenomena.

In the first chapter we motivated the need for PCM and talked briefly about some prevalent uncertainty analysis techniques. In Chapter 2 we introduced the one-dimensional PCM, and discussed its underlying theory of Gaussian quadrature and orthogonal polynomials. We illustrated one-dimensional PCM with the help of an example from physical chemistry and wrapped up the chapter by discussing some refinements to the algorithm with sensitivity information and error bounds on PCM. In Chapter 3 we provided our generalization of PCM to handle multiple, correlated uncertain parameters. We also proposed a way to mesh PCM with boundary load flow algorithms for filtering out some of the uncertain variables and an order-selection algorithm for selecting the appropriate PCM order. We provided examples to illustrate each idea. In Chapter 4 we talked about optimization problems and discussed the possibility of using PCM for solving them. This aspect of PCM is in the incipient stage, but the results look promising.

Information-theoretic approaches for reducing the number of input uncertain variables were discussed in Chapter 5. Two cases were identified and the approach for each was discussed with illustrative examples. In Chapter 6, we applied PCM to model the input-output relationship for a 14 Bus IEEE test system. Then with the aid of the input variable reduction techniques discussed earlier, we developed another mapping for the same system with far fewer input variables. The results obtained were promising.

7.1 Future Directions

- *Optimization:* As mentioned earlier, our attempt at applying PCM for solving optimization problems is rudimentary. In the future, we would like to attune PCM for handling problems in this domain.

- *Justification:* We have corroborated most of our results with analytical proofs. In the future we would like to make refinements, if necessary, and attempt to publish our results in an applied math context.
- *Software Package:* Develop a package for PCM. We do have Mathematica modules for PCM and most of the results presented in this report were generated using them. However, we would want a version of PCM coded in a programming language like C/C++.

7.2 Areas of Application

This report shows a glimpse of PCM's versatility. In the future, we would like to expand PCM applications to areas that have not been discussed in this report. In the power systems context, these might include prediction of relay operation (or mis-operation), in studies for which the measured impedance from a relay location is the key output quantity of interest.

8. Project-Related Publications

Hockenberry, J.R. and B.C. Lesieutre. “Evaluation of Uncertainty in Dynamic Simulations of Power System Models: The Probabilistic Collocation Method,” *IEEE Transactions on Power Systems*, Volume 19, August 2004, pp. 1483-1491.

Ramamurthy, D., "Smart Simulation Techniques for the Evaluation of Parametric Uncertainties in Black Box Systems,” M.S. Thesis, Computer Science, supervisor: S. Roy, Washington State University, May 2005.

Roy, S., D. Ramamurthy, and B.C. Lesieutre, “Studies on the Probabilistic Collocation Method and Its Application to Power System Analysis,” *Proceedings of the 36th Annual North American Power Symposium,* Moscow, Idaho, Aug. 2004

Roy, S., G.C. Verghese, and B.C. Lesieutre, “Moment Linear Stochastic Systems,” presented at INCINCO, August 2004.

Y. Wan, S. Roy, B. Lesieutre, "Uncertainty evaluation through mapping identification in intensive dynamic simulations," *IEEE Transactions on Systems, Man, and Cybernetics*, Part A, pp. 1094-1104, vol. 40, no. 5, Sept. 2010.

9. References

[1] J. R. Hockenberry and B. C. Lesieutre, "Evaluation of uncertainty in dynamic simulations of power systems models: the probabilistic collocation method," *IEEE Transactions on Power Systems*, pp. 1483-1491, v. 19, no. 3, Aug. 2004.

[2] J. R. Hockenberry, *Evaluation of Uncertainties in Dynamic, Reduced-Order Power System Models,*, Ph.D. Thesis, Massachusetts Institute of Technology, Sep., 2000.

[3] S. Roy, D. Ramamurthy, and B.C. Lesieutre, "Studies on the probabilistic collocation method and its application to power systems analysis," *Proceedings of the 36th Annual North American Power Symposium,* Moscow, Idaho, Aug. 2004

[4] R. Billinton, M. Fotuhi-Firuzabad, and L. Bertling, "Bibliography on the application of probability *methods* in power system reliability evaluation 1996-1999," *IEEE Transactions on Power Systems*, Vol. 16, No. 4, Nov. 2001.

[5] M. Webster, M.A. Tatang, and G. J. McRae, "Application of the probabilistic collocation method for an uncertainty analysis of a simple ocean model," Joint Program on the Science and Policy of Global Change, MIT, Cambridge, MA, Tech. Rep. 4, Jan. 1996.

[6] T. B. Nguyen, M. A. Pai, and I. A. Hiskens, "Bounding uncertainty in power system dynamic simulations", *Proceedings of the IEEE PES 2000 Winter Meeting*, Singapore, January 2000.

[7] A. Dimitrovski and K. Tomsovic, "Boundary load flow solutions", *IEEE Transactions on Power Systems*, Vol. 19, No. 1, Feb. *2004*, pp. 348-355.

[8] P.J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, 2nd Edition, Academic Press, New York, 1984.

[9] G. Akrivis, "The Error Norm of Certain Gaussian Quadrature Formulae", *Mathematics of Computation*, Vol. 45, No. 172, Oct. 1985, pp. 513-519.

[10] A. R. Bergen and V. Vittal, *Power Systems Analysis (2nd Edition)*, Prentice Hall, Upper Saddle River New Jersey, 2000.

[11] A. M. L. da Silva and V. L. Arienti, "Probabilistic load flow by a multilinear simulation algorithm", *Proceedings of the IEE C*, Vol. 137, No. 4, pp. 276-282, July 1990.

[12] R. N. Allan and A. M. L. da Silva, "Probabilistic load flow using multilinearizations", *Proceedings of the IEE C,* Vol. 128, No. 5, pp. 280-287, Sep. 1981.

[13] T. M. Cover and J. Thomas, *Elements of Information Theory*, Wiley & Sons, New York, 1991.

[14] S.S. Isukapalli and P. G. Georgopoulos, "Uncertainty analysis of transport transformation models," PhD. Thesis, Rutgers State University, Jan., 1999.

[15] S. S. Isukapalli, A. Roy, and P. G. Georgopoulos, "Stochastic Response Surface Methods (SRSMs) for Uncertainty Propagation: Application to Environmental and Biological Systems," *Risk Analysis, 18(3):351-363, 1998*.

[16] R. Y. Rubenstein, *Simulation and the Monte Carlo method*, Wiley series in probability and statistics, 1981.

[17] L. Mathelin and M.Y. Hussaini, " A Stochastic collocation algorithm for uncertainty analysis,"

[18] D. P. Bertsekas, *Nonlinear Programming, 2nd Edition*, Athena Scientific, Bellmont, 1999.

[19] L. Mathelin and M.Y. Hussaini, "A Stochastic Collocation algorithm for uncertainty analysis," NASA/CR-2003-212153, 2003.

[20] E. Toth and A. Brath, "Flood forecasting using artificial neural networks in black-box and conceptual rainfall-runoff modeling," *Proceedings of the 1st Biennial Meeting of the iEMSs* 2002

[21] I. Rivals and L. Personnaz, "Black-box modeling with state-space neural networks," *Neural Adaptive Control Technology,* R. Zbikowski and K. J. Hunt eds., World Scientific 1996, pp. 237-264.

[22] http://en.wikipedia.org/wiki/Ideal_gas, *Wikipedia*, the free encyclopedia.

[23] http://mathews.ecs.fullerton.edu/n2003/gaussianquad/GaussianQuadBib/Links/GaussianQuadBib_lnk_3.html, Bibliography for Gauss-Legendre quadrature.

[24] http://mathworld.wolfram.com/CorrelationCoefficient.html, *Mathworld*, Wolfram Research.

[25] A. Markoff, "Sur la methode de Gauss pur le Calcul approche des integrals," *Math. Ann., 25,* pp. 427-432, 1885.

[26] M. A. Bokhari, H. Taufiq, "An extension of Gauss quadrature formula," *Technical Report 319,* Dept. of Mathematical Sciences, King Fahd University, May 2004.

[27] J. Luthi, G. Haring, "Bottleneck Analysis for Computer and Communication Systems with Workload Variabilities & Uncertainties," in *Proceedings of the*

Second International Symposium on Mathematical Modeling, I. Troch and F. Breitenecker, ed., pp 525-534, Vienna, Austria, Feb. 1997.

[28] D. O'Neill and Y. Li, "Optimal routes and flows in multicasting over ad hoc networks," in *Proceedings of the IEEE Vehicular Technologies Conference*, May 2004, Milan, Italy.

[29] D. Gross and C. M. Harris, *Fundamentals of Queueing Theory, 2nd ed*. John Wiley and Sons, 1985.

[30] F. P. Kelly, *Reversibility and Stochastic Networks, 3rd ed.* John Wiley and Sons, 1994. Available at www.statslab.cam.ac.uk/~frank.rsn.html.